Automated Proofs of Combinatorial Identities

Yuriy Choliy

August 10, 2004

1 Definitions

Hypergeometric identity:

$$\sum_{k=-\infty}^{\infty} F(n,k) = \text{rhs}(n),$$

where both $\frac{F(n+1,k)}{F(n,k)}$ and $\frac{F(n,k+1)}{F(n,k)}$ are rational functions of n and k.

Standard form for hypergeometic series:

$${}_r\mathrm{F}_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z \right] \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k k!} z^k, \tag{1.1}$$

where

$$(a)_k \stackrel{\text{def}}{=} a(a+1)(a+2)\cdots(a+k-1);$$
$$(a)_0 = 1,$$

so $k!$ can be written as $(1)_k$.

From (1.1), the ratio

$$\frac{F(n,k+1)}{F(n,k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_r)}{(k+b_1)(k+b_2)\cdots(k+b_s)(k+1)} z.$$

2 Examples

Many well-known series, including those that are encountered in elementary calculus courses, are hypergeometric. For instance, McLauren series for e^x:

Example 1.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The ratio $F(x,k+1)/F(x,k) = \frac{x}{k+1}$, is clearly a rational function of x and k, so we can write:

$$e^x = {}_0\mathrm{F}_0\left[\begin{matrix}-\\-\end{matrix};x\right]$$

Here are two more examples:

Example 2.

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} \frac{F(n,k+1)}{F(n,k)} \\ &= \frac{-x^2}{(2k+2)(2k+3)} = \frac{-x^2}{4}\frac{1}{(k+1)(k+3/2)},\end{aligned}$$

so

$$\sin x = x\ \ {}_0\mathrm{F}_1\left[\begin{matrix}-\\3/2\end{matrix};\frac{-x^2}{4}\right].$$

Example 3.

$$\begin{aligned}\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x\sum_{k=0}^{\infty} \frac{(-1)^k x^(2k)}{2k+1} \frac{F(n,k+1)}{F(n,k)} \\ &= \frac{-x^2(2k+1)}{2k+3}\frac{k+1}{k+1},\end{aligned}$$

so

$$\arctan x = x\ \ {}_2\mathrm{F}_1\left[\begin{matrix}1/2,1\\3/2\end{matrix};-x^2\right].$$

3 The WZ Method

The WZ method is explained in [1, Chapter 7]. Consider

$$\sum_k \tilde{F}(n,k) = \mathrm{rhs}(n).$$

Let $F(n,k) = \tilde{F}(n,k)/\mathrm{rhs}(n)$. Then

$$\sum_k F(n,k) = 1 \tag{3.1}$$

(3.1) can be proven by finding a rational function $R(n,k)$ such that

$$F(n+1,k) - F(n,k) = R(n,k+1)F(n,k+1) - R(n,k)F(n,k) \tag{3.2}$$

Using (3.2) when $k = -1$, we can find that $R(n,0) = 0$. Solving (3.2) for $R(n,k+1)$, we obtain:

$$R(n,k+1) = \frac{F(n,k)}{F(n,k+1)}\left[\frac{F(n+1,k)}{F(n,k)} + R(n,k) - 1\right]. \tag{3.3}$$

From (3.3), we can find $R(n,1)$, $R(n,2)$ and so on.
Then we get a system of equations:

$$\begin{aligned}
0 &= a_0 \\
(b_0+b_1+b_2+b_3)R(n,1) &= a_0+a_1+a_2+a_3 \\
(b_0+2b_1+4b_2+8b_3)R(n,2) &= a_0+2a_1+4a_2+8a_3 \\
(b_0+3b_1+9b_2+27b_3)R(n,3) &= a_0+3a_1+9a_2+27a_3 \\
&\vdots
\end{aligned}$$

4 Preliminary results

We tested our method using machine arc051 in Busch computer lab on the following classical hypergeometric summation formulas, all of which can be found in [2, Appendix III, pp. 243–245]:

Theorem 1 (Gauss).

$${}_2\mathrm{F}_1\left[\begin{matrix}a,b\\c\end{matrix};1\right] = \frac{(c-1)!(c-a-b-1)!}{(c-a-1)!(c-b-1)!} \tag{4.1}$$

Theorem 2 (Gauss).

$${}_2\mathrm{F}_1\left[\begin{matrix}a,b\\(1+a+b)/2\end{matrix};1/2\right] = \frac{(-1/2)!((a-b-1)/2)!}{((a-1)/2)!((b-1)/2)!} \tag{4.2}$$

Theorem 3 (Chu-Vandermonde).

$${}_2\mathrm{F}_1\left[\begin{matrix}a,-n\\c\end{matrix};1\right] = \frac{(c-a)_n}{(c)_n} \tag{4.3}$$

Theorem 4 (Pfaff-Saalschuetz).

$${}_3\mathrm{F}_2\left[\begin{matrix}a,b,-n\\c,a+b-c-n+1\end{matrix};1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \tag{4.4}$$

Theorem 5 (Kummer).

$${}_2\mathrm{F}_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};-1\right] = \frac{(a-b)!(a/2)!}{a!(a/2-b)!} \tag{4.5}$$

Theorem 6.

$${}_4\mathrm{F}_3\left[\begin{matrix}a,1+a/2,b,c\\a/2,1+a-b,1+a-c\end{matrix};-1\right] = \frac{(a-b)!(a-c)!}{a!(a-b-c)!} \tag{4.6}$$

Theorem 7 (Bailey).

$$ {}_2\mathrm{F}_1\left[\begin{matrix} a, 1-a \\ c \end{matrix}; \right] = \frac{(c/2-1)!(c/2-1/2)!}{(c/2+a/2-1)!}(c/2-a/2-1/2)! \tag{4.7} $$

Theorem 8 (Dixon).

$$ {}_3\mathrm{F}_2\left[\begin{matrix} a, b, -n \\ 1+a-b, 1+a+n \end{matrix}; 1\right] = \frac{(1+a)_n(1+a/2-b)_n}{(1+a/2)_n(1+a-b)_n} \tag{4.8} $$

Theorem 9.

$$ {}_4\mathrm{F}_3\left[\begin{matrix} a, 1+a/2, b, -n \\ a/2, 1+a-b, 1+a+n \end{matrix}; -1\right] = \frac{(1+a)_n}{(1+a-b)_n} \tag{4.9} $$

Theorem 10.

$$ {}_3\mathrm{F}_2\left[\begin{matrix} 1+a/2, a, -n \\ a/2, b \end{matrix}; 1\right] = (b-a-n-1)\frac{(b-a)_{n-1}}{(b)_n} \tag{4.10} $$

Identity	Our time (sec)	EKHAD's time (sec)	ratio
(4.1)	0.0156	0.1898	12.167
(4.2)	0.0068	0.2367	34.81
(4.3)	0.0184	0.2358	12.82
(4.4)	0.5814	0.5546	0.9539
(4.5)	0.0164	0.2681	16.348
(4.6)	0.2717	0.5342	1.966
(4.7)	0.1035	0.2559	2.472
(4.8)	0.0877	0.3921	4.471
(4.9)	0.1759	0.5613	2.850
(4.10)	0.905	0.220	0.243

References

[1] M. Petrovšek, H. S. Wilf, D. Zeilberger, *A=B*, Wellesley, MA, A. K. Peters, 1996.

[2] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.