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Note

Geometric drawings of K_n with few crossings

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Abstract

We give a new upper bound for the rectilinear crossing number $\overline{cr}(n)$ of the complete geometric graph K_n. We prove that $\overline{cr}(n) \leqslant 0.380559\binom{n}{4} + \Theta(n^3)$ by means of a new construction based on an iterative duplication strategy starting with a set having a certain structure of halving lines.

Keywords: Rectilinear crossing number; Crossing number; Geometric graph; Complete graph; Convex quadrilateral

1. Introduction

The *crossing number* $cr(G)$ of a simple graph G is the minimum number of edge crossings in any drawing of G in the plane, where each edge is a simple curve. The *rectilinear crossing number* $\overline{cr}(G)$ is the minimum number of edge crossings in any *geometric drawing* of G, that is a drawing of G in the plane where the vertices are points in general position and the edges are straight segments. The crossing numbers have many applications to Discrete Geometry and Computer Science, see, for example, [9] and [10].

In this paper we contribute to the problem of determining $\overline{cr}(K_n)$, where K_n denotes the complete graph on n vertices. Specifically, we construct geometric drawings of K_n with a small number of crossings. For simplicity we write $\overline{cr}(n) = \overline{cr}(K_n)$. We note that a geometric drawing of K_n is determined by the location of its vertices and two edges cross each other if and only if the quadrilateral determined by their vertices is convex. Thus our drawings also provide constructions of n-element point sets with small number of convex quadrilaterals determined by the n points. The problem of finding the asymptotic behavior of $\overline{cr}(n)$ is also important because of its

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close relation to Sylvester's four point problem [13]: Determine the probability that four points selected uniformly at random on a given domain, form a convex quadrilateral. It is easy to see that $\{\overline{cr}(n)/\binom{n}{4}\}$ is a non-decreasing sequence bounded above by 1. Thus $v = \lim_{n\to\infty} \overline{cr}(n)/\binom{n}{4}$ exists, and Scheinerman and Wilf [12] proved that v is the infimum, over all open sets R with finite area, of the probability that four randomly chosen points in R are in convex position.

When this problem was first investigated, the best lower bounds were obtained by using an averaging argument over subsets whose crossing numbers were known. For example, it is well known that $\overline{cr}(5) = 1$, so by counting the crossings generated by every subset of size 5 it is easy to get $\overline{cr}(n) \geqslant (1/5)\binom{n}{4}$. Wagner [14] was the first to use a different approach, he proved $v \geqslant 0.3288$. Then the authors [1] and independently Lovász et al. [8], used allowable sequences to prove $v \geqslant 3/8 = 0.375$. Lovász et al. [8] managed to even improve this v by 10^{-5}, and Balogh and Salazar [5] refined this technique even further to obtain the currently best bound of $v \geqslant 0.37533$.

The history of the improvements on the upper bounds is as follows: In the early seventies Jensen [7] and Singer [11] obtained $v \leqslant 7/18 < 0.3888$, and $v \leqslant 5/13 < 0.38462$, respectively. Much later Brodsky et al. [6] constructed point sets with nested non-concentric triangles yielding $v \leqslant 6467/16{,}848 < 0.383844$. Then Aichholzer et al. [3] devised a lens-replacement construction depending on a suitable initial set. They obtained $v \leqslant 0.380739$ and later on, Aichholzer and Krasser found a better initial set [4], which gave the previously known best bound of $v \leqslant 0.38058$.

In this paper we prove the following theorem.

Theorem 1. $\overline{cr}(n) \leqslant \frac{29{,}969}{78{,}750}\binom{n}{4} + \Theta(n^3) < (0.380559)\binom{n}{4} + \Theta(n^3)$.

This theorem is based on the following stronger result which may improve the upper bound in the future. To accomplish this we would only need a point-set P having a halving-line matching (defined next) and with a better $\binom{n}{4}$ coefficient.

Theorem 2. *If P is a N-element point set in general position, with N even, and P has a halving-line matching*; *then*

$$\overline{cr}(n) \leqslant \left(\frac{24\,cr(P) + 3N^3 - 7N^2 + (30/7)N}{N^4}\right)\binom{n}{4} + \Theta\left(n^3\right).$$

Let $P = \{p_1, p_2, \ldots, p_N\}$ be a general position point set in the plane. We define a *halving line* of P as a line passing through two points in P and leaving the same number of points of P on either side of the line. According to this, N needs to be even for P to have a halving line.

Consider a bipartite graph $G = (P, H)$ where H is the set of halving lines of P and $p \in P$ is adjacent to $l \in H$ if p is on the line l. If there is a matching for P in the graph G we say that this matching is a *halving-line matching* of P. Note that such a matching induces a function on P, $p_i \mapsto p_{f(i)}$, such that $p_i p_{f(i)}$ is a halving line of P and if $i \neq j$ then $p_i p_{f(i)}, p_j p_{f(j)}$ are different halving lines.

All known optimal drawings of K_n with n even $6 \leqslant n \leqslant 16$ have a halving-line matching. The same is true for all the best known drawings with $18 \leqslant n \leqslant 48$ reported in [2]. The only exception seems to be P_4, the set consisting of a triangle with a point inside. This set achieves $\overline{cr}(4) = 0$ but it only has three halving lines, so a halving-line matching is impossible.

The previously best general constructions, obtained by Aichholzer et al. [3], used a lens-replacement construction yielding

$$\overline{cr}(n) \leqslant \left(\frac{24\,cr(P) + 3N^3 - 7N^2 + 6N}{N^4}\right)\binom{n}{4} + \Theta\left(n^3\right) \tag{1}$$

for any initial point set P with N elements, N even. This bound requires the same number of points in each of the lenses. They further refined their method by considering lenses of different sizes. Unfortunately this post-optimization depends heavily on the structure of the initial set and it seems that the improved constant cannot be verified other than by computer calculations. In [4], Aichholzer and Krasser use a set of 54 points with 115,999 crossings which gives $\overline{cr}(n) \leqslant (0.380601)\binom{n}{4} + \Theta(n^3)$ using (1) and they state the bound $\overline{cr}(n) \leqslant (0.38058)\binom{n}{4} + \Theta(n^3)$ after the post-optimization process (the details about the lens sizes are not available in [4]). Note that our construction from Theorem 2 always gives a better bound compared to (1) as long as the starting set P has a halving-line matching.

2. The construction

The construction is based on the following lemma.

Lemma 3. *If P is a N-element point set, N even, and P has a halving-line matching; then there is a point set $Q = Q(P)$ in general position, $|Q| = 2N$, Q also has a halving-line matching, and $\overline{cr}(Q) = 16\,\overline{cr}(P) + (N/2)(2N^2 - 7N + 5)$.*

Proof. Q is constructed as follows. Each point $p_i \in P$ will be replaced by a pair of points q_{i1} and q_{i2}. Using the function f induced by the halving-line matching we define

$$q_{i1} = p_i + \varepsilon\frac{p_{f(i)} - p_i}{\|p_{f(i)} - p_i\|} \quad \text{and} \quad q_{i2} = p_i - \varepsilon\frac{p_{f(i)} - p_i}{\|p_{f(i)} - p_i\|},$$

where ε is small enough so that all points q_{j1}, q_{j2} coming from a point p_j located to the left (right) of $\overrightarrow{p_i p_{f(i)}}$; are also located to the left (right) of the lines $q_{i1}q_{f(i)1}$ and $q_{i1}q_{f(i)2}$. In other words, the cone with vertex q_{i1} spanned by the segment $q_{f(i)1}q_{f(i)2}$ does not contain any of the points q_{jx} with $j \neq i, f(i)$. One way to find such an ε is the following: suppose that P is contained in a disk with diameter D, since P is in general position there is $\delta > 0$ so that the strips of width δ centered at the lines $p_i p_{f(i)}$ have only the points p_i and $p_{f(i)}$ with P in common. The portion contained in the disk of any cone with center in the line $p_i p_{f(i)}$, axis $p_i p_{f(i)}$, and with an opening angle of $2\arctan(\delta/2D)$; is a subset of the strip of width δ around $p_i p_{f(i)}$. Thus we can start with a small arbitrary value $\varepsilon_0 < \delta$ such that all points q_{j1}, q_{j2} are inside the disk. Obtain ε_i as the distance from $p_{f(i)}$ to the lines subtending the cone with center q_{i1}, axis $p_i p_{f(i)}$, and angle $2\arctan(\delta/2D)$. And finally set ε as the minimum of all the ε_i.

From the halving-line matching definition we deduce that no two points in P are associated to the same halving line. This, together with the choice of ε, guarantees that Q is in general position whenever P is in general position. By construction the line $q_{i1}q_{i2}$ is a halving line of Q. In addition, since q_{i1} is in the interior of the triangle $q_{i2}, q_{f(i)1}, q_{f(i)2}$ the line $q_{i1}q_{f(i)1}$ (and also the line $q_{i1}q_{f(i)2}$) is a halving line of Q. Then

$$\left\{\left(q_{i1}, \overleftrightarrow{q_{i1}q_{f(i)1}}\right)\colon i = 1, 2, \ldots, N\right\} \cup \left\{\left(q_{i2}, \overleftrightarrow{q_{i1}q_{i2}}\right)\colon i = 1, 2, \ldots, N\right\}$$

is a halving-line matching of Q.

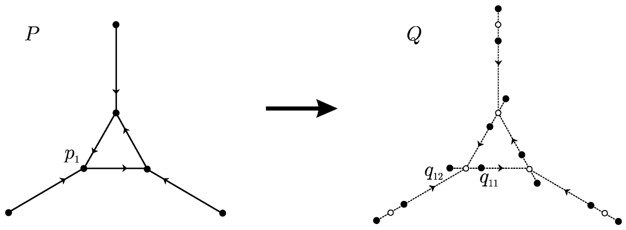

Fig. 1. Construction of the set Q.

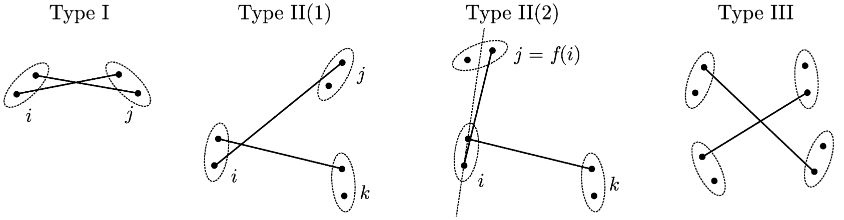

Fig. 2. Different types of crossings.

Now we proceed to calculate $\overline{cr}(Q)$ by counting crossings according to three different types. This method of counting was originally used by Aichholzer et al. [3], to calculate the number of crossings in their lens-replacement construction.

Type I Two points in pair i and two in pair $j \neq i$. There are $\binom{N}{2}$ ways of choosing pairs i and j and all of them determine a crossing except when $j = f(i)$ or $i = f(j)$. Since there are exactly N pairs $(i, f(i))$, the total number of crossings in this case is $\binom{N}{2} - N$.

Type II Two points in pair i and the two other in pairs j and k all pairs distinct. First there are N choices for the pair i. Then we have two cases, when $j, k \neq f(i)$, and when either j or k equals $f(i)$. In the first case (see Fig. 2, Type II(1)), to have a crossing, both p_j and p_k must be on the same side of the line $q_{i1}q_{i2}$. Thus there are $2\binom{N/2-1}{2}$ ways of choosing j and k and then 4 choices for the second indices x and y for the points q_{jx} and q_{ky}. In the second case (see Fig. 2, Type II(2)) we can assume $j = f(i)$. Then there are 2 choices for the second index x of q_{jx}. Again, to have a crossing, we need q_{jx} and p_k on the same side of $q_{i1}q_{i2}$. So there are $N/2 - 1$ ways of choosing k and 2 choices for the second index y of q_{ky}. The total number of Type II crossings is $N(8\binom{N/2-1}{2} + 4(N/2 - 1))$.

Type III Each point in a different pair. To have a crossing each of the four pairs must come from a crossing in P, so there are $\overline{cr}(P)$ possible pairs, and there are 2 choices for the second index in each pair. Thus there are $16\,\overline{cr}(P)$ number of crossings of Type III.

The list of crossings types is complete since by construction none of the segments $q_{i1}q_{i2}$ participate in any crossing. Adding Types I–III yields the result. □

The strength of this lemma can be easily seen when it is applied to the set P_6 that minimizes $\overline{cr}(6) = 3$. We obtain $\overline{cr}(12) \leqslant \overline{cr}(Q) = 153$ which happens to be the correct value of $\overline{cr}(12)$ (a drawing is shown in Fig. 1). It took some effort [3] to find this point set in the past. In addition, if we use the sets P_{30}, P_{36}, P_{48} obtained by Aichholzer et al., and reported in web page [2],

Table 1
Improved bounds for 60, 72, and 96 points

n	$\overline{cr}(P_n)$	$\overline{cr}(Q(P_n))$	Old bound for $\overline{cr}(2n)$	New bound for $\overline{cr}(2n)$
30	9726	179,541	$\leqslant$ 179,554	$\leqslant$ 179,541
36	21,175	381,010	$\leqslant$ 381,020	$\leqslant$ 381,010
48	71,028	1,239,096	$\leqslant$ 1,239,139	$\leqslant$ 1,239,096

we obtain sets P_{60}, P_{72}, P_{96} with less number of crossings than those previously known (see Table 1).

3. Proof of Theorems 1 and 2

Let $S_0 = P$ and $S_{k+1} = Q(S_k)$ for $k \geqslant 0$, where $Q(S_k)$ is the set given by Lemma 3. We first prove by induction that for $k \geqslant 0$,

$$\overline{cr}(S_k) = 16^k\,\overline{cr}(P) + N^3 8^{k-1}\bigl(2^k - 1\bigr) - \frac{7}{6}N^2 4^{k-1}\bigl(4^k - 1\bigr) + \frac{5}{14}N 2^{k-1}\bigl(8^k - 1\bigr). \tag{2}$$

This identity is trivially true when $k = 0$. By Lemma 3 we have that $|S_k| = 2^k N$ and

$$\begin{aligned}\overline{cr}(S_{k+1}) &= 16\,\overline{cr}(S_k) + \frac{|S_k|}{2}\bigl(2|S_k|^2 - 7|S_k| + 5\bigr)\\ &= 16\,\overline{cr}(S_k) + N^3 8^k - \frac{7}{2}N^2 4^k + \frac{5}{2}N 2^k.\end{aligned}$$

Then by induction hypothesis (2) we get

$$\begin{aligned}\overline{cr}(S_{k+1}) &= 16^{k+1}\,\overline{cr}(P) + N^3 8^k\bigl(2^{k+1} - 2\bigr) - \frac{7}{6}N^2 4^k\bigl(4^{k+1} - 4\bigr) + \frac{5}{14}N 2^k\bigl(8^{k+1} - 8\bigr)\\ &\quad + N^3 8^k - \frac{7}{2}N^2 4^k + \frac{5}{2}N 2^k\\ &= 16^{k+1}\,\overline{cr}(P) + N^3 8^k\bigl(2^{k+1} - 1\bigr) - \frac{7}{6}N^2 4^k\bigl(4^{k+1} - 1\bigr) + \frac{5}{14}N 2^k\bigl(8^{k+1} - 1\bigr),\end{aligned}$$

which proves (2) for all $k \geqslant 0$. Letting $n = |S_k| = 2^k N$ identity (2) becomes

$$\overline{cr}(n) \leqslant \overline{cr}(S_k) = \left(\frac{24\,cr(P) + 3N^3 - 7N^2 + (30/7)N}{24N^4}\right)n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{5}{28}n. \tag{3}$$

This proves Theorem 2 for $n = 2^k N$. To establish the result for general n we first show that $\{\overline{cr}(n)/\binom{n}{4}\}$ is an increasing sequence. Indeed, if $|A| = n$ and $\overline{cr}(n) = \overline{cr}(A)$ then

$$(n-4)\,\overline{cr}(n) = (n-4)\,\overline{cr}(A) = \sum_{\substack{B\subseteq A\\ |B|=n-1}} \overline{cr}(B) \geqslant n \cdot \overline{cr}(n-1)$$

and consequently $\overline{cr}(n)/\binom{n}{4} \geqslant \overline{cr}(n-1)/\binom{n-1}{4}$.

Suppose now that $2^k N \leqslant n < 2^{k+1}N$. From (3) we obtain $\overline{cr}(2^{k+1}N) \leqslant c\binom{2^{k+1}N}{4} + c_1(2^{k+1}N)^3$ where c is the coefficient of $\binom{n}{4}$ in Theorem 2 and c_1 is constant. Then

$$\overline{cr}(n) \leqslant \frac{\overline{cr}(2^{k+1}N)\binom{n}{4}}{\binom{2^{k+1}N}{4}} \leqslant c\binom{n}{4} + 8c_1\bigl(2^k N\bigr)^3 \leqslant c\binom{n}{4} + 8c_1 n^3$$

Table 2
Coordinates of point set P_{30} with 9776 crossings

ith point $= p_i = (x\text{-coordinate}, y\text{-coordinate})$			
$p_1 = (9259, 16{,}598)$	$p_9 = (28{,}477, 16{,}613)$	$p_{17} = (5141, 23{,}755)$	$p_{25} = (9075, 15{,}320)$
$p_2 = (9763, 16{,}199)$	$p_{10} = (15{,}909, 16{,}415)$	$p_{18} = (9154, 17{,}055)$	$p_{26} = (7921, 13{,}407)$
$p_3 = (9977, 16{,}397)$	$p_{11} = (9446, 15{,}905)$	$p_{19} = (0, 32{,}394)$	$p_{27} = (5206, 8451)$
$p_4 = (10{,}248, 16{,}225)$	$p_{12} = (9540, 16{,}541)$	$p_{20} = (6820, 20{,}921)$	$p_{28} = (9121, 15{,}603)$
$p_5 = (10{,}666, 16{,}385)$	$p_{13} = (9262, 16{,}627)$	$p_{21} = (9949, 16{,}415)$	$p_{29} = (480, 0)$
$p_6 = (12{,}849, 16{,}335)$	$p_{14} = (9282, 16{,}947)$	$p_{22} = (9355, 16{,}177)$	$p_{30} = (6432, 10{,}637)$
$p_7 = (18{,}577, 16{,}451)$	$p_{15} = (8912, 17{,}261)$	$p_{23} = (9419, 15{,}893)$	
$p_8 = (10{,}391, 16{,}281)$	$p_{16} = (7842, 19{,}232)$	$p_{24} = (9146, 15{,}771)$	

Table 3
Halving-line matching of P_{30}

$a, (a, b)$ means $(p_a, p_a p_b)$			
1, (1, 3)	9, (9, 13)	17, (17, 18)	25, (25, 26)
2, (2, 4)	10, (10, 11)	18, (18, 19)	26, (26, 27)
3, (3, 7)	11, (11, 12)	19, (19, 21)	27, (27, 28)
4, (4, 5)	12, (12, 14)	20, (20, 21)	28, (28, 29)
5, (5, 8)	13, (13, 14)	21, (21, 23)	29, (29, 27)
6, (6, 7)	14, (14, 15)	22, (22, 23)	30, (30, 29)
7, (7, 9)	15, (15, 16)	23, (23, 24)	
8, (8, 10)	16, (16, 20)	24, (24, 25)	

which proves Theorem 2. Now, to prove Theorem 1, consider the set $P = P_{30}$ with coordinates in Table 2 obtained from [2]. It satisfies that $\overline{cr}(P_{30}) = 9726$ and it can be verified that the set of point-line pairs in Table 3 represents a halving-line matching of P_{30}. We then get

$$\overline{cr}(n) \leqslant \frac{29{,}969}{78{,}750}\binom{n}{4} + \Theta\left(n^3\right).$$

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