1 Introductory Remarks

Five sections comprise the bulk of these notes. The first introduces the notion of a vertex operator algebra. Meanwhile, the second section deals with the notion of a module in vertex operator algebra theory. In the third, we introduce Zhu's algebra while the fourth explains the $C_2$-cofiniteness condition. The $C_2$-conjecture is introduced in the final section.

Before we proceed, we should mention a few notational conventions we use throughout these notes. First, for any vector space $V$, we define $V[[x]]$ to be the space of formal power series with coefficients in $V$; that is,

$$V[[x]] = \left\{ \sum_{n \in \mathbb{N}} v_n x^n \mid v_n \in V \right\},$$

where $\mathbb{N}$ is the set of non-negative integers.

With that said, we define $V[[x, x^{-1}]]$ to be the space of formal Laurent series with coefficients in $V$:

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V \right\}.$$

Next, consider an element $v(x) \in V[[x, x^{-1}]]$, where $V$ is some arbitrary vector space. By $\lim_{x \to a} v(x)$, we simply mean formally substituting $x$ by $a$, where $a$ could be from our vector space $V$ (if $V$ is an algebra, for instance), its underlying field $F$, or perhaps even another ring so long as the substitution "makes sense". So, $\lim_{x \to a} v(x) = v(x)|_{x=a}$. The limit may or may not exist, and one should always verify the limit’s existence when utilising the limit notation. Please consult [FLM] or [LL] for more on these matters.

For $v(x) \in V[[x, x^{-1}]]$, we define the formal residue operator $\text{Res}_x$ with respect to the formal variable $x$ by

$$\text{Res}_x v(x) = \text{the coefficient of } x^{-1} \text{ in } v(x).$$

Finally, we have the "delta-function" $\delta(x)$,

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$
2 Vertex Operator Algebras

In the literature, a vertex operator algebra is sometimes denoted by the quadruple \((V, Y, 1, \omega)\). The complete axiomatic data of a vertex operator algebra, however, is as follows.

A vertex operator algebra is a \(\mathbb{Z}\)-graded vector space \(V\) over a field \(F\),
\[
V = \prod_{n \in \mathbb{Z}} V(n),
\]
equipped with a linear map
\[
Y : V \otimes V \rightarrow V[[x, x^{-1}]],
\]
or equivalently, a linear map
\[
Y(\cdot, x) : V \rightarrow \left(\text{End}(V)\right)[[x, x^{-1}]]
\]
with \(Y(v, x)\) usually expressed as \(\sum_{n \in \mathbb{Z}} v_n x^{-n-1}\), where \(v_n \in \text{End}(V)\) for each \(n \in \mathbb{Z}\).

In this vector space \(V\), we have two distinguished homogeneous elements \(1 \in V(0)\) and \(\omega \in V(2)\). The vector \(1\) is known as the vacuum vector while the vector \(\omega\) is called the conformal vector. The following list of axioms complete the notion of a vertex operator algebra.

For all \(u, v \in V\), we have:

- \(\dim V(n) < \infty\) for all \(n \in \mathbb{Z}\), and \(V(n) = 0\) for \(n\) sufficiently negative;
- with \(Y(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1}\), we have \(u_n v = 0\) for \(n\) sufficiently positive; in other words, for each \(v \in V\), the Laurent series \(Y(u, x)v \in V[[x, x^{-1}]]\) is truncated from below;
- the vacuum property, which states \(Y(1, x) = 1_V\), the identity on \(V\);
- the creation property, \(Y(v, x)1 \in V[[x]]\) and \(\lim_{x \to 0} Y(v, x)1 = v\);
- the Jacobi identity,
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2);
\]
- with \(Y(\omega, x) = \sum_{n \in \mathbb{Z}} \omega_n x^{-n-1} = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}\), we have the Virasoro algebra relations,
\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c_V,
\]
for \(m, n \in \mathbb{Z}\), where \(c_V \in F\) is known as the central charge;
• for a homogeneous \( v \in V_{(n)} \), we have \( L(0)v = nv \);

• and finally, the \( L(-1) \)-derivative property, which states

\[
Y \left( L(-1)v, x \right) = \frac{d}{dx} Y(v, x).
\]

So much for axiomatic basics. Note, a vector space \( V \), which is not necessarily graded, with a “vertex operator” \( Y : V \otimes V \rightarrow V[[x, x^{-1}]] \) and a vacuum vector \( \mathbf{1} \), which altogether satisfy the second to fifth bulleted axioms, is known as a vertex algebra. We will be focusing our attention solely on vertex operator algebras.

Here is an example of a vertex operator as found in section 1.2 of [LL]; the reader is invited to explore detailed expositions on these matters in books and papers such as [FLM], [LL], or [LW].

Consider the space

\[
A = \mathbb{C}[y_{\frac{1}{2}}, y_{\frac{3}{2}}, y_{\frac{5}{2}}, \ldots]
\]

of polynomials in infinitely many commuting formal variables \( y_n, n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \). Then,

\[
Y = \exp \left( \sum_{n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots} y_n \frac{x^n}{n} \right) \exp \left( -2 \sum_{n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots} \frac{\partial}{\partial y_n} x^{-n} \right)
\]

is an example of a vertex operator on the space \( A \).

Here, \( \exp \) is the formal exponential series and \( x \) is another formal variable that commutes with the \( y_n \). The \( y_n \) and \( \frac{\partial}{\partial y_n} \) within the arguments of the exponentials are viewed as operators on \( A \). The operator \( y_n \) corresponds to multiplication by \( y_n \) while the operator \( \frac{\partial}{\partial y_n} \) corresponds to partial differentiation with respect to the variable \( y_n \) in \( A \); these operators are known as creation and annihilation operators, respectively, in the literature. Furthermore, each coefficient \( Y_j \) of \( x^j \) in \( Y \) is a well-defined linear operator on \( A \). In fact, we have the following result:

**Theorem** [LW] The operators

\[
1, \ y_n, \ \frac{\partial}{\partial y_n}, \text{ and } Y_j
\]

span a Lie algebra of operators acting on \( A \). Moreover, this Lie algebra is precisely a copy of the affine Lie algebra \( \mathfrak{sl}(2) \).
3 Vertex Operator Algebra Modules

We introduce the notion of a vertex operator algebra module in this section. Let $V$ be a vertex operator algebra with vertex operator $Y$, vacuum vector $1$, and conformal vector $\omega$. A vertex operator algebra module, or $V$-module for short, is a $\mathbb{C}$-graded vector space $W$,

$$W = \prod_{h \in \mathbb{C}} W(h),$$

equipped with a linear map

$$Y_W : V \otimes W \to W[[x, x^{-1}]],$$

or equivalently a linear map

$$Y_W(\cdot, x) : V \to \left(\text{End}(W)\right)[[x, x^{-1}]]$$

where

$$v \mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}.$$

We require “all the defining properties of a vertex operator algebra that make sense hold” for a $V$-module. That is, for $u, v \in V$ and $w \in W$, the following properties are satisfied:

- $\dim W(h) < \infty$ for all $h \in \mathbb{C}$;
- $W(h) = 0$ for $h \in \mathbb{C}$ with sufficiently negative real parts;
- with $Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} \omega_n x^{-n-1} = \sum_{n \in \mathbb{Z}} L_W(n)x^{-n-2}$, we have
  $$L_W(0)w = hw$$
  for homogeneous $w \in W(h)$;
- $Y_W(v, x)w$ is truncated from below; so, $v_n w = 0$ for $n$ sufficiently positive;
- the vacuum property, $Y_W(1, x) = 1_W$, the identity map on $W$;
- and the Jacobi identity,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{x_0}\right)Y_W(v, x_2)Y_W(u, x_1)$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_W\left(Y(u, x_0)v, x_2\right).$$
4 Zhu’s algebra

In this section, we introduce an associative algebra \( A(V) \) commonly known as Zhu’s algebra from a vertex operator algebra \( V \). Zhu’s algebra is crucial to the study of the representation theory of vertex operator algebras. One major property the algebra \( A(V) \) enjoys is the one-to-one correspondence between isomorphism classes of the irreducible \( A(V) \)-modules and the isomorphism classes of the irreducible \( V \)-modules.

Fix a vertex operator algebra \((V,Y,1,\omega)\).

**Definition** For homogeneous \( v \in V_{(n)} \), we define the weight of \( v \) to be \( n \), which we denote \( wt v = n \).

Note, this means for all homogeneous \( v \in V_{(n)} \), we have

\[
L(0)v = nv = (wt v)v.
\]

We define a bilinear operation \( * \) on \( V \) as follows. For homogeneous \( v \in V \) and for \( w \in V \),

\[
v * w = \text{Res}_x \left( \frac{(1 + x)^{wt v}}{x} \right) Y(v,x)w.
\]

We denote by \( O(V) \) the subspace of \( V \) spanned by elements of the form

\[
\text{Res}_x \left( \frac{(1 + x)^{wt v}}{x^2} \right) Y(v,x)w
\]

for all homogeneous \( v \in V \) and all \( w \in V \).

Take \( A(V) = V/O(V) \). That is, \( A(V) \) is the quotient space of our vertex operator algebra \( V \) by \( O(V) \). We have the following theorem of Zhu:

**Theorem** [Zhu] \( O(V) \) is a two-sided ideal for the bilinear operation \( * \) and so, \( * \) is defined on \( A(V) \). Moreover, \( A(V) \) is an associative algebra under the multiplication \( * \); the image of the vacuum \( 1 \) in \( A(V) \) is the unit of the algebra \( A(V) \); the image of \( \omega \) is in the center of \( A(V) \); and \( A(V) \) has a filtration

\[
A_0(V) \subset A_1(V) \subset \ldots,
\]

where \( A_n(V) \) is the image of \( \bigoplus_{0 \leq i \leq n} V_{(i)} \).

\[\blacksquare\]
5 The $C_2$-condition

Along with the associative algebra $A(V)$, Zhu introduced what is now known as the $C_2$-cofiniteness condition in [Z1] and [Z2] (in these papers, the $C_2$-cofiniteness condition is called “Condition C”, and it is defined slightly differently from how we define it here). Zhu used the $C_2$-cofiniteness condition to establish the modular invariance of the space of characters for a vertex operator algebra. Since then, there have been other “cofiniteness conditions” generalising the $C_2$-condition made by those working in the theory, but we will not be using any of these generalisations.

Let $(V,Y,1,\omega)$ be a vertex operator algebra. Let $C_2(V)$ be the subspace of $V$ spanned by elements of the form

\[ u_{-2}v \]

for all $u, v \in V$, where the general notation $u_n v$ refers to the coefficient of $x^{-n-1}$ in $Y(u,x)v = \sum_{n \in \mathbb{Z}} (u_n v)x^{-n-1}$.

The vertex operator algebra $V$ satisfies the $C_2$-cofiniteness condition if the quotient space $V/C_2(V)$ is finite-dimensional. In this case, we say that $V$ is $C_2$-cofinite.

6 The $C_2$-conjecture

At last, we arrive at the kernel of these notes, the $C_2$-conjecture:

**Conjecture** A vertex operator algebra $V$ is $C_2$-cofinite if $A(V)$ is finite-dimensional.

In general, it is known that $A(V)$ is finite-dimensional whenever the vertex operator algebra $V$ is $C_2$-cofinite. So, the conjecture may be stated: “A vertex operator algebra $V$ is $C_2$-cofinite if and only if $A(V)$ is finite-dimensional.”
References


