Standard Affine Lie Algebra Modules, Vertex Operator Algebras, and the Function $\Delta(H, x)$

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Outline

- Affine Lie algebra background
- Changing the vertex operator algebra module actions with $\Delta(H, x)$
- Results
A vector space (over \( \mathbb{C} \)) \( g \) is called a \textbf{Lie algebra} if \( g \) is equipped with a bilinear map \( [\cdot, \cdot] : g \times g \to g \) such that:
A vector space (over $\mathbb{C}$) $\mathfrak{g}$ is called a Lie algebra if $\mathfrak{g}$ is equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that:

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Notice, this looks like the Jacobi identity if we rewrite it:

\[ x \cdot (y \cdot v) = [x, y] \cdot v + y \cdot (x \cdot v) \]
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- Finite dimensional simple Lie algebras are “built” up from Cartan matrices.
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Now, let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra with Cartan matrix \( C \).
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  We call these elements fundamental roots.
- We can pick a basis $\Pi^\vee = \{H_1, \ldots, H_\ell\}$ for $\mathfrak{h}$ such that for all $1 \leq i, j \leq \ell$, we have that $\alpha_i(H_j) = a_{ji}$. 

Standard Affine Lie Algebra Modules, Vertex Operator Algebras, and the Function $\Delta(H, x)$ – p. 6
We fix the basis \( \{\lambda_1, \ldots, \lambda_\ell\} \) which is dual to \( \Pi^\vee \) (i.e. \( \lambda_i(H_j) = \delta_{i,j} \)).
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- Define the fundamental coweights \( \{ H^{(1)}, \ldots, H^{(\ell)} \} \subset \mathfrak{h} \) to be the basis dual to \( \{ \alpha_1, \ldots, \alpha_\ell \} \).
The Affinization of $\mathfrak{g}$

Define $\hat{\mathfrak{g}}$ as follows:

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

where $c$ is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m \langle a, b \rangle \delta_{m+n,0}c$$

for every $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

$\hat{\mathfrak{g}}$ is the (untwisted) affine Lie algebra associated with $\mathfrak{g}$. 
Given $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can define a linear functional $(k, \lambda) \in \hat{\mathfrak{h}}^* = (\mathfrak{h} \oplus \mathbb{C}c)^*$ by the following:

- For all $h \in \mathfrak{h}$, let $(k, \lambda)(h) = \lambda(h)$.
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Recall that \( \lambda_i \ (1 \leq i \leq \ell) \) are the fundamental weights of \( g \). For convenience, let \( \lambda_0 = 0 \). Then, define \( k\Lambda_i = (k, \lambda_i) \) for \( 0 \leq i \leq \ell \).
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Likewise, the "nice" (what are called standard or highest weight integrable) irreducible modules for $\hat{\mathfrak{g}}$ are determined by their highest weights $\Lambda = \sum_{i=0}^{l} m_i \Lambda_i$ where $m_i \in \mathbb{Z}_{\geq 0}$
Standard Modules

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- Likewise, the “nice” (what are called standard or highest weight integrable) irreducible modules for $\hat{\mathfrak{g}}$ are determined by their highest weights $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i$ where $m_i \in \mathbb{Z}_{\geq 0}$.

- By $L(\Lambda)$, we mean the irreducible highest weight $\hat{\mathfrak{g}}$-module with highest weight $\Lambda$. 

Standard Affine Lie Algebra Modules, Vertex Operator Algebras, and the Function $\Delta(H, x)$ – p. 10
Theorem: Let $k \in \mathbb{Z}_{>0}$. Then, $L(k\Lambda_0) = L(k, 0)$ has the structure of a simple vertex operator algebra.
The Simple VOA $L(k\Lambda_0)$

**Theorem:** Let $k \in \mathbb{Z}_{>0}$. Then, $L(k\Lambda_0) = L(k, 0)$ has the structure of a simple vertex operator algebra. Also, it’s modules (VOA modules) are exactly the standard modules for $\hat{g}$ of level $k$ (i.e. $c \cdot v = kv$).
\( \Delta(H, x) \) and Li’s Theorem

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Define $\Delta(H, x) = x^{H(0)} \exp \left( \sum_{k=1}^{\infty} \frac{H(k)}{-k} (-x)^{-k} \right)$, where $H(k)$ denotes the action of $H \otimes t^k$. 
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**Theorem [Li]:** For any irreducible \( L(k\Lambda_0) \)-module \( W \), \( W^{(H)} = (W, Y_W(\Delta(H, x) \cdot, x) \) is also an irreducible \( L(k\Lambda_0) \)-module.
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**Theorem [Li]:** For any irreducible $L(k\Lambda_0)$-module $W$, $W^{(H)} = (W, Y_W(\Delta(H, x), x))$ is also an irreducible $L(k\Lambda_0)$-module.

In fact, if $H = \sum_{i=1}^{\ell} m_i H_i$, $(m_i \in \mathbb{Z})$, then $W$ is isomorphic to $W^{(H)}$ as an $L(k\Lambda_0)$-module.
It’s not hard to see that

$$\Delta(H' + H'', x) = \Delta(H', x) \Delta(H'', x)$$

and

$$\Delta(0, x) = Id$$
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A Property of $\Delta(H, x)$

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$$\Delta(H' + H'', x) = \Delta(H', x)\Delta(H'', x)$$

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- So, if we know what each $H^{(j)}$ does, we know everything!
- Let’s see what each the $H^{(j)}$’s do.
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Method

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We examined simple Lie algebras by their types, starting with $A_n$, i.e. algebras of type $sl(n + 1, \mathbb{C})$. 
Weight Diagram for $A_2$

A weight diagram
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- For $A_2$, the case of $sl_3(\mathbb{C})$, we have $L^{(H^{(1)})}(k, a\lambda_1 + b\lambda_2)$ is isomorphic to $L(k, (k - a - b)\lambda_1 + a\lambda_2)$ and $L^{(H^{(2)})}(k, a\lambda_1 + b\lambda_2)$ is isomorphic to $L(k, b\lambda_1 + (k - a - b)\lambda_2)$. 
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- We also worked out the case for $A_3 = \mathfrak{sl}_4(\mathbb{C})$. 

Standard Affine Lie Algebra Modules, Vertex Operator Algebras, and the Function $\Delta(H, x)$ – p. 16
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- We also have an idea of how to generalize for $A_l$, but are yet to determine if it will be fruitful.
We found the following

- Fortunately, the cases of the simple Lie algebras $G_2$, $F_4$ and $E_8$ turned out to be trivial, as the coroot and coweight lattices were equal, so, by Li’s Theorem, $\Delta$ takes each module back to itself.
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We also worked out the case for $D_4$, but it is not listed here, and we have not yet found a pattern for type $D$. 