Outline

Classical Algebras: associative algebras and Lie algebras

Nonclassical Algebras: vertex algebras

Affine Lie algebras and modules

Changing the vertex operator algebra module actions with $\Delta(H, x)$

Results
Classical Algebras

Associative Algebras

and

Lie Algebras
Vector Space (over a field $\mathbb{F}$): 

A **vector space**, $V$, over a field $\mathbb{F}$ is a set equipped with two operations: vector addition and scalar multiplication such that...

- **Associative** $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
- **Identity** There exists $0 \in V$ such that $v + 0 = v = 0 + v$ for all $v \in V$.
- **Inverses** For each $v \in V$ there exists $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.
- **Commutative** $u + v = v + u$ for all $u, v \in V$.
- **Distributive** $c(u + v) = cu + cv$ for all $u, v \in V$ and $c \in \mathbb{F}$.
- **Distributive** $(a + b)v = av + bv$ for all $v \in V$ and $a, b \in \mathbb{F}$.
- **Associative** $(ab)v = a(bv)$ for all $v \in V$ and $a, b \in \mathbb{F}$.
- **Identity** $1v = v$ for all $v \in V$.

**Examples:**

$\mathbb{R}^3$ over $\mathbb{R}$, $\mathbb{C}^3$ over $\mathbb{C}$, or $\mathbb{R}[x]$ over $\mathbb{R}$. 
Algebras (over $\mathbb{F}$):

An Algebra, $\mathcal{A}$, is a vector space (over some field $\mathbb{F}$) equipped with a multiplication map: $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $m$ is \textit{bilinear}.

- Instead of writing $m(u, v)$, we usually use juxtaposition to denote multiplication: $m(u, v) = uv$
- \textbf{bilinear} means: for all $u, v, w \in \mathcal{A}$ and $c \in \mathbb{F}$...
  Linear on the Left \hspace*{1em} $(u + v)w = uw + vw$ and $(cu)v = c(uv)$
  Linear on the Right \hspace*{1em} $u(v + w) = uv + uw$ and $u(cv) = c(uv)$
Examples:

Real Matrix Algebras  Let $\mathcal{A} = \mathbb{R}^{n \times n}$ be all $n \times n$ matrices with real entries.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix}$$

Polynomial Algebras  Let $\mathcal{A} = \mathbb{R}[x, y]$ be all polynomials in two indeterminants ($x$ and $y$) with real coefficients.

$$(3x^2 + xy + 2)(2y - x) = 6x^2y - 3x^3 + 2xy^2 - x^2y + 4y^2 - 2x$$

Cross Product Algebra  Let $\mathcal{A} = \mathbb{R}^3$ here we multiply vectors by taking their cross product: $u \times v$.

$$\langle 1, 2, 3 \rangle \times \langle 1, 0, -1 \rangle = \det \begin{pmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} = \langle -2, 4, -2 \rangle$$
Special Properties:

Let \( \mathcal{A} \) be an algebra (over some field \( \mathbb{F} \)).

**Associative** \( \mathcal{A} \) is **Associative** if \( u(vw) = (uv)w \) for all \( u, v, w \in \mathcal{A} \).

**Unital** \( \mathcal{A} \) is **Unital** or an algebra with **identity** if there exists some \( 1 \in \mathcal{A} \) such that \( 1v = v1 = v \) for all \( v \in \mathcal{A} \).

**Commutative** \( \mathcal{A} \) is **Commutative** if \( uv = vu \) for all \( u, v \in \mathcal{A} \).

**Examples:**

- \( \mathcal{A} = \mathbb{R}[x, y] \) is a commutative associative unital algebra. Its identity is the polynomial 1.
- \( \mathcal{A} = \mathbb{R}^{n \times n} \) is an associative unital algebra, but it is *not* commutative (unless \( n = 1 \)). Its identity is the *identity matrix* \( I_n \).
- \( \mathcal{A} = \mathbb{R}^3 \) equipped with the cross product is a non-associative non-commutative algebra and has no identity. So what kind of algebra is this?
Lie Algebras

Let $L$ be an algebra (over some field $\mathbb{F}$). Instead of using juxtaposition, let’s denote multiplication with a bracket: $m(u, v) = [u, v]$. $L$ is called a Lie Algebra if the following axioms hold:

**Skew-Commutative** \([v, v] = 0\) for all \(v \in L\).

**Jacobi Identity** \([[u, v], w] + [[v, w], u] + [[w, u], v] = 0\) for all \(u, v, w \in L\).

**Examples:**

- \(A = \mathbb{R}^3\) equipped with the cross product is a Lie algebra. Remember that \(v \times v = 0\) (the cross product of parallel vectors is zero). A tedious calculation shows that the Jacobi identity holds as well.

- If we give the matrix algebra $\mathbb{R}^{n \times n}$ a different multiplication, called the **commutator bracket**, defined by $[A, B] = AB - BA$, it becomes a Lie algebra. To remind ourselves that we are using a different “multiplication” we call this algebra $\mathfrak{gl}(n, \mathbb{R})$. 
What do the axioms really say?

- The first axiom, \([v, v] = 0\), implies the following:

\[
0 = [u+v, u+v] = [u, u+v] + [v, u+v] = [u, u] + [u, v] + [v, u] + [v, v] = [u, v] + [v, u]
\]

Therefore, \([u, v] = -[v, u]\) (almost commutative!)

- The Jacobi identity says much more. Using the above property we can re-write the Jacobi identity as follows:

\[
[[u, v], w] + [[v, w], u] + [[w, u], v] = 0
\]
Lie Algebras

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Lie Algebras

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  Therefore, \([u, v] = -[v, u]\) (almost commutative!)

- The Jacobi identity says much more. Using the above property we can re-write the Jacobi identity as follows:

  \[ [[u, v], w] - [v, [w, u]] = [u, [v, w]] \]
What do the axioms really say?

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$$0 = [u + v, u + v] = [u, u + v] + [v, u + v] = [u, u] + [u, v] + [v, u] + [v, v] = [u, v] + [v, u]$$

Therefore, $[u, v] = -[v, u]$ (almost commutative!)

• The Jacobi identity says much more. Using the above property we can re-write the Jacobi identity as follows:

$$[[u, v], w] + [v, -[w, u]] = [u, [v, w]]$$
What do the axioms really say?

- The first axiom, $[v, v] = 0$, implies the following:

$$0 = [u+v, u+v] = [u, u+v] + [v, u+v] = [u, u] + [u, v] + [v, u] + [v, v] = [u, v] + [v, u]$$

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- The Jacobi identity says much more. Using the above property we can re-write the Jocobi identity as follows:

$$[[u, v], w] + [v, [u, w]] = [u, [v, w]]$$
What do the axioms really say?

The Jacobi identity says much more. Using the above property we can re-write the Jacobi identity as follows:

\[
[u, [v, w]] = [[u, v], w] + [v, [u, w]]
\]

\[
\frac{d}{dt} [f(t)g(t)] = \frac{d}{dt} [f(t)] g(t) + f(t) \frac{d}{dt} [g(t)]
\]

Let \( \mathcal{A} \) be an algebra and \( \partial : \mathcal{A} \to \mathcal{A} \). If \( \partial(uv) = \partial(u)v + u\partial(v) \) for all \( u, v \in \mathcal{A} \), then we say that \( \partial \) is a derivation of \( \mathcal{A} \).

The Jacobi identity simply says, “The multiplication operators of \( L \) are derivations.”
Non-Classical Algebras:

Vertex (Operator) Algebras
**Origins of Vertex Operator Algebras:**


1980s Mathematicians use vertex operators to study certain representations of affine Lie algebras.

1984 I. Frenkel, J. Lepowsky, and A. Meurman construct $\mathcal{V}^\natural$.

1986 R. Borcherds introduces a set of axioms for a notion which he calls a “vertex algebra”.

The Definition of a Vertex Algebra

Notation:

- $V[x]$ polynomials in $x$ with coefficients in $V$.
- $V[x, x^{-1}]$ Laurent polynomials in $x$ with coefficients in $V$.
- $V[[x]]$ power series in $x$ with coefficients in $V$.
- $V((x))$ lower truncated Laurent series with coefficients in $V$.
- $V[[x, x^{-1}]]$ Laurent series in $x$ with coefficients in $V$.

WARNING: $\mathbb{C}[[x, x^{-1}]]$ is not an algebra! Sometimes multiplication isn’t well defined.

Example: $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ is the formal delta function. Notice that $(\delta(x))^2$ is undefined.
Algebras (over $\mathbb{C}$):
Let $V$ be a vector space over $\mathbb{C}$.

Vertex Algebras:
Let $V$ be a vector space over $\mathbb{C}$. 
**Algebras (over \( \mathbb{C} \)):**
Equip \( V \) with a bilinear map \( \cdot : V \times V \to V \) (a multiplication map).
\[(u, v) \mapsto u \cdot v\]

**Vertex Algebras:**
Equip \( V \) with **infinitely many** bilinear maps \( n : V \times V \to V \) (where \( n \in \mathbb{Z} \)).
\[(u, v) \mapsto u_n v\]
The Definition of a Vertex Algebra

**Algebras (over \( \mathbb{C} \)):**
Equip \( V \) with a bilinear map \( \cdot : V \times V \to V \) (a multiplication map).
\[
(u, v) \mapsto u \cdot v
\]

**Vertex Algebras:**
Equip \( V \) with a bilinear map
\[
Y(\cdot, x) : V \times V \to V[[x, x^{-1}]]
\]
\((V[[x, x^{-1}]] \text{ are Laurent series with coefficients in } V)\).
\[
(u, v) \mapsto Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v x^{-n-1}
\]
The Definition of a Vertex Algebra

**Algebras (over \( \mathbb{C} \)):**
Equip \( V \) with a bilinear map \( \cdot : V \times V \rightarrow V \) (a multiplication map).
\[(u, v) \mapsto u \cdot v\]

**Vertex Algebras:**
Equip \( V \) with a bilinear map
\[Y(\cdot, x) : V \times V \rightarrow V((x))\]
\((V((x))) \) are lower truncated Laurent series with coefficients in \( V \).
\[(u, v) \mapsto Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v x^{-n-1}\]
where \( u_n v = 0 \) for \( n \gg 0 \).
Algebras (over $\mathbb{C}$):
To be a Lie algebra $V$’s multiplication must be skew-symmetric and satisfy the Jacobi identity.
\[(uv)w + (vw)u + (wu)v = 0\]

Vertex Algebras:
$V$’s vertex operators must satisfy the Jacobi identity.

\[x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)w\]

\[= x_2^{-1}\delta \left(\frac{x_1 - x_0}{-x_2}\right) Y(Y(u, x_0)v, x_2)w\]
The Definition of a Vertex Algebra

**Algebras (over \(\mathbb{C}\)):**
To be a Lie algebra \(V\)'s multiplication must be skew-symmetric and satisfy the Jacobi identity.

\[
(ux)x - x(ux) + x(xu) = 0
\]

\[
u(\mathfrak{v}w) = (uv)w + v(\mathfrak{u}w) \quad \text{(the product rule)}
\]

**Vertex Algebras:**
\(V\)'s vertex operators must satisfy the Jacobi identity.

\[
x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) \mathcal{Y}(u, x_1)(\mathcal{Y}(v, x_2)w) =
\]

\[
x^{-1}_2 \delta \left( \frac{x_1 - x_0}{-x_2} \right) \mathcal{Y}(\mathcal{Y}(u, x_0)v, x_2)w + x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(v, x_2)(\mathcal{Y}(u, x_1)w)
\]
Algebras (over $\mathbb{C}$):
To be a unital algebra $V$ must have an identity vector $1$.

\[ 1u = u1 = u \]

Vertex Algebras:
A vertex algebra has a vaccuum vector $1$.

\[ Y(1, x)u = u \quad \text{and} \quad Y(u, x)1 = e^{x \mathcal{D}}u \]

In particular, $Y(u, 0)1 = u$. 
The Definition of a Vertex Algebra

**Algebras (over \( \mathbb{C} \)):**

\( V \) is a commutative algebra if...

\[ uv = vu \]

**Vertex Algebras:**

Vertex operators satisfy a property called **locality**.

\[(x_1-x_2)^N Y(u, x_1)Y(v, x_2) = (x_1-x_2)^N Y(v, x_2)Y(u, x_1)\]

for some \( N \gg 0 \).
Algebras (over $\mathbb{C}$):

$V$ is an associative algebra if...

$$(uv)w = u(vw)$$

Vertex Algebras:

Vertex operators satisfy a property called weak associativity.

$$(x_1 - x_2)^N Y(Y(u, x_1)v, x_2)w = (x_1 + x_2)^N Y(u, x_1 + x_2)(Y(v, x_2)w)$$

for some $N \gg 0$
Holomorphic Vertex Algebras

Let $\mathcal{A}$ be a commutative associative unital algebra (whose identity is 1) and let $D : \mathcal{A} \rightarrow \mathcal{A}$ be a derivation of $\mathcal{A}$. Define $Y(u, x) = e^{xD}u$. That is:

$$Y(u, x)v = \sum_{n=0}^{\infty} u_{-n-1}vx^n$$

where $u_{-n-1}v = \frac{1}{n!}D^n(u)v$

Then, $\mathcal{A}$ equipped with this vertex operator map, $Y(\cdot, x)$, becomes a vertex algebra with vacuum vector 1.

Note: If $D = 0$, we simply have $Y(u, x)v = uv$. That is: All commutative associative unital algebras are vertex algebras.
Define a matrix $C = (a_{ij})_{1 \leq i, j \leq \ell}$. 
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We call \( C \) a **Cartan matrix** iff:
Cartan Matrices

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- For all $i, j$, $a_{ij} = 0$ iff $a_{ji} = 0$
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- $C$ is a positive definite matrix

Now, let $g$ be a finite dimensional simple Lie algebra with Cartan matrix $C$. 
Simple Lie algebras have a “nice” decomposition. We pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. 
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- We can pick a basis $\Pi = \{\alpha_1, ..., \alpha_\ell\}$ for the dual space $\mathfrak{h}^*$.

We call these elements fundamental roots.
Simple Lie algebras have a “nice” decomposition. We pick a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \).

- The dimension of \( \mathfrak{h} \) is \( \ell \). We call this the rank of \( \mathfrak{g} \).
- We can pick a basis \( \Pi = \{\alpha_1, \ldots, \alpha_\ell\} \) for the dual space \( \mathfrak{h}^* \).
  We call these elements fundamental roots.
- We can pick a basis \( \Pi^\vee = \{H_1, \ldots, H_\ell\} \) for \( \mathfrak{h} \) such that for all \( 1 \leq i, j \leq \ell \), we have that \( \alpha_i(H_j) = a_{ji} \).
We fix the basis \( \{\lambda_1, ..., \lambda_\ell\} \) which is dual to \( \Pi^\vee \) (i.e. \( \lambda_i(H_j) = \delta_{i,j} \)).
Notation:

- We fix the basis \( \{ \lambda_1, ..., \lambda_\ell \} \) which is dual to \( \Pi^\vee \) (i.e. \( \lambda_i(H_j) = \delta_{i,j} \)).
- These \( \lambda_i \)'s are called fundamental weights.
We fix the basis \( \{ \lambda_1, ..., \lambda_\ell \} \) which is dual to \( \Pi^\vee \) (i.e. \( \lambda_i(H_j) = \delta_{i,j} \)).

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Define the fundamental coweights \( \{ H^{(1)}, \ldots, H^{(\ell)} \} \subset \h \) to be the basis dual to \( \{ \alpha_1, ..., \alpha_\ell \} \).
Define $\hat{g}$ as follows:

$$\hat{g} := g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

where $c$ is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m \langle a, b \rangle \delta_{m+n,0} c$$

for every $a, b \in g$ and $m, n \in \mathbb{Z}$. $\hat{g}$ is the (untwisted) affine Lie algebra associated with $g$. 

Vertex Operator Algebra Structure of Standard Affine Lie Algebra Modules – p. 33/43
Given $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can define a linear functional $(k, \lambda) \in \hat{\mathfrak{h}}^* = (\mathfrak{h} \oplus \mathbb{C}c)^*$ by the following:

- For all $h \in \mathfrak{h}$, let $(k, \lambda)(h) = \lambda(h)$.
- On $\mathbb{C}c$, let $(k, \lambda)(c) = k$. 


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Recall that $\lambda_i \ (1 \leq i \leq \ell)$ are the fundamental weights of $\mathfrak{g}$. For convenience, let $\lambda_0 = 0$. Then, define $k\Lambda_i = (k, \lambda_i)$ for $0 \leq i \leq \ell$. 
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$\Lambda_0 = (1, \lambda_0) = (1, 0)$, $\Lambda_1 = (1, \lambda_1)$, $\ldots$, $\Lambda_\ell = (1, \lambda_\ell)$ are the fundamental weights for $\hat{\mathfrak{g}}$. 
Irreducible Representations for $\mathfrak{g}$ are determined by their highest weights ($\approx$ eigenvalue for a special “highest weight vector”).
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If our module is to be finite dimensional, then the highest weight $\lambda = \sum_{i=1}^{l} m_i \lambda_i$ where $m_i \in \mathbb{Z}_{\geq 0}$. 
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Likewise, the “nice” (what are called standard or highest weight integrable) irreducible modules for $\hat{\mathfrak{g}}$ are determined by their highest weights $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i$ where $m_i \in \mathbb{Z}_{\geq 0}$.
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By $L(\Lambda)$, we mean the irreducible highest weight $\hat{\mathfrak{g}}$-module with highest weight $\Lambda$. 
The Simple VOA $L(k\Lambda_0)$

**Theorem:** Let $k \in \mathbb{Z}_{>0}$. Then, $L(k\Lambda_0) = L(k, 0)$ has the structure of a simple vertex operator algebra.
Theorem: Let $k \in \mathbb{Z}_{>0}$. Then, $L(k\Lambda_0) = L(k,0)$ has the structure of a simple vertex operator algebra. Also, its modules (VOA modules) are exactly the standard modules for $\hat{\mathfrak{g}}$ of level $k$ (i.e. $c \cdot v = kv$).
Let $H = \sum_{i=1}^{\ell} m_i H^{(i)}$ where $m_i \in \mathbb{Z}$. 
Let \( H = \sum_{i=1}^{\ell} m_i H^{(i)} \) where \( m_i \in \mathbb{Z} \).

Define \( \Delta(H, x) = x^{H(0)} \exp \left( \sum_{k=1}^{\infty} \frac{H(k)}{-k} (-x)^{-k} \right) \),

where \( H(k) \) denotes the action of \( H \otimes t^k \).
**Δ(H, x) and Li’s Theorem**

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where \( H(k) \) denotes the action of \( H \otimes t^k \).

**Theorem [Li]:** For any irreducible \( L(k\Lambda_0) \)-module \( W \), \( W^{(H)} = (W, Y_W(\Delta(H, x) \cdot, x) \) is also an irreducible \( L(k\Lambda_0) \)-module.
Let $H = \sum_{i=1}^{\ell} m_i H^{(i)}$ where $m_i \in \mathbb{Z}$.

Define $\Delta(H, x) = x^{H(0)} \exp \left( \sum_{k=1}^{\infty} \frac{H(k)}{-k} (-x)^{-k} \right)$, where $H(k)$ denotes the action of $H \otimes t^k$.

**Theorem [Li]:** For any irreducible $L(k\Lambda_0)$-module $W$, $W^{(H)} = (W, Y_W(\Delta(H, x) \cdot, x)$ is also an irreducible $L(k\Lambda_0)$-module.

In fact, if $H = \sum_{i=1}^{\ell} m_i H_i$, $(m_i \in \mathbb{Z})$, then $W$ is isomorphic to $W^{(H)}$ as an $L(k\Lambda_0)$-module.
It’s not hard to see that

\[ \Delta(H' + H'', x) = \Delta(H', x) \Delta(H'', x) \]

and

\[ \Delta(0, x) = Id \]
It’s not hard to see that

\[ \Delta(H' + H'', x) = \Delta(H', x)\Delta(H'', x) \]

and

\[ \Delta(0, x) = Id \]

So, if we know what each \( H^{(j)} \) does, we know everything!
A Property of $\Delta(H, x)$

- It’s not hard to see that

$$\Delta(H' + H'', x) = \Delta(H', x)\Delta(H'', x)$$

and

$$\Delta(0, x) = Id$$

- So, if we know what each $H^{(j)}$ does, we know everything!

- Let’s see what each the $H^{(j)}$’s do.
While the $\Delta$ plays nicely with the VOA structure, it does some strange things to the $\hat{g}$-module structure.
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Method

- While the $\Delta$ plays nicely with the VOA structure, it does some strange things to the $\hat{g}$-module structure.

- To determine which module we ended up with, we searched for the "new" highest weight vectors.

- We examined simple Lie algebras by their types, starting with $A_n$, i.e. algebras of type $sl(n + 1, \mathbb{C})$. 
For $A_1$, the case of $sl_2(\mathbb{C})$, we have $L^{(H^{(1)})}(k, n)$ is isomorphic to $L(k, k - n)$. 
We found the following

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  $L^{(H^{(1)})}(k, n)$ is isomorphic to $L(k, k - n)$.

- For $A_2$, the case of $sl_3(\mathbb{C})$, we have
  $L^{(H^{(1)})}(k, a\lambda_1 + b\lambda_2)$ is isomorphic to
  $L(k, (k - a - b)\lambda_1 + a\lambda_2)$ and
  $L^{(H^{(2)})}(k, a\lambda_1 + b\lambda_2)$ is isomorphic to
  $L(k, b\lambda_1 + (k - a - b)\lambda_2)$. 
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- We have worked out the cases for all finite dimensional simple Lie algebras.
We found the following

Fortunately, the cases of the simple Lie algebras $G_2$, $F_4$ and $E_8$ turned out to be trivial, as the coroot and coweight lattices were equal, so, by Li’s Theorem, $\Delta$ takes each module back to itself.
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- Fortunately, the cases of the simple Lie algebras $G_2$, $F_4$ and $E_8$ turned out to be trivial, as the coroot and coweight lattices were equal, so, by Li’s Theorem, $\Delta$ takes each module back to itself.

- For each simple Lie algebra, we obtained “nice” Weyl group elements that determine the new highest weight vector our $\Delta$ gives us. Unfortunately, we had to do it case by case for each type, and have yet to find a pattern that will work for any simple Lie algebra.
The End