Some Definitions and Notes

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1 Lie Algebras

We start by defining a Lie algebra \( g \):

Definition 1.1. Let \( g \) be a vector space over an arbitrary field \( F \) equipped with a product \([,] : g \times g \to g\). We call \( g \) a Lie algebra iff, for all \( x, y, z \in g \) and for all \( a, b \in F \):

\[
[x, y] = ax + by, z = a[x, y] + b[x, z] \quad (1.1)
\]

\[
[x, x] = 0 \quad (1.2)
\]

\[
[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad (the \ Jacobi \ Identity) \quad (1.3)
\]

An immediate result we get if the characteristic of \( F \neq 2 \) is:

\[
[x, y] = -[y, x] \quad (1.4)
\]

Proof:

\[0 = [x+y, x+y] = [x, x+y] + [y, x+y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]\]

The result follows. ■

Now, for some examples of Lie algebras:

Example: The vector space of \( n \times n \) matrices over a field \( F \), \( M_{n \times n}(F) \), with \([,] : M_{n \times n}(F) \times M_{n \times n}(F) \to M_{n \times n}(F) \) defined as the commutator:

\[
[A, B] = AB - BA \quad (1.5)
\]

As a Lie algebra, \( M_{n \times n}(F) \) is often referred to as \( gl(n, F) \).

Equivalently, let \( V \) be a vector space, and consider the vector space \( End(V) \). We can make \( End(V) \) into a Lie algebra via:

\[
[f, g] = f \circ g - g \circ f \quad (1.6)
\]

where \( \circ \) is function composition. This Lie algebra is often referred to as \( gl(V) \).

Example: The vector space \( \mathbb{R}^3 \), with \([,] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) given by the cross product:

\[
[u, v] = u \times v \quad (1.7)
\]

We need also the natural notion of a Lie subalgebra:
Definition 1.2. We call a subspace $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra iff $\mathfrak{h}$ is closed under $[\cdot, \cdot]$.

We also need the natural notion of a Lie algebra homomorphism:

Definition 1.3. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras. A linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{g}'$ is called a Lie algebra homomorphism iff:

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

If $\phi$ is a bijection, then we call $\phi$ an isomorphism.

2 Tensor Product

For those unfamiliar with a tensor product, I’ll give a definition:

Definition 2.1. Let $U$ and $V$ be vector spaces over a field $F$. We define their cartesian product to be

$$U \times V = \{(u, v) | u \in U, v \in V\} \quad (2.1)$$

Now, let $I$ be the subspace of $U \times V$ generated by the elements:

$$(u_1 + u_2, v) - (u_1, v) - (u_2, v)$$
$$(u, v_1 + v_2) - (u, v_1) - (u, v_2)$$
$$(cu, v) - c(u, v)$$
$$(u, cv) - c(u, v)$$

Definition 2.2. We define the tensor product of $U$ and $V$ to be:

$$(U \times V) / I \quad (2.2)$$

and denote it $U \otimes V$.

So, $U \otimes V$ is just the quotient space, and its cosets are denoted $u \otimes v$ for $u \in U$ and $v \in V$. $U \otimes V$ is itself a vector space over the field $F$.

For a more intuitive idea of what is actually going on here, we are basically taking the vector space $U \times V$ and imposing the following relations:

$$(u_1 + u_2, v) = (u_1, v) + (u_2, v)$$
$$(u, v_1 + v_2) = (u, v_1) + (u, v_2)$$
$$(cu, v) = c(u, v)$$
$$(u, cv) = c(u, v)$$

and calling it $U \otimes V$. 

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3 Lie Algebra Representations

Let V be a vector space.

Definition 3.1. A Lie algebra homomorphism $\pi : g \rightarrow gl(V)$ is called a representation of g, and V is called the representation space.

One of the most common examples of a representation is called the adjoint representation:

$$ad : g \rightarrow gl(g)$$

which is given by

$$ad(x) = ad_x = [x, \cdot]$$

That is, $ad_x(y) = [x, y]$.

4 Lie Algebra Modules

Now, we examine modules for Lie algebras (i.e., spaces that the Lie algebra has a certain kind of ‘action’ on).

Definition 4.1. Let V be a vector space. We call V a g-module iff there exists a bilinear map $\cdot : g \times V \rightarrow V$ such that:

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

One can imagine this looks sort of like the Jacobi Identity, and also sort of like commutator (I prefer to remember the latter).

The notion of a representation of a Lie algebra is equivalent to the notion of module for a Lie algebra. If $\pi$ is a representation on V, then V can be made into a g-module, given $x \in g$, $v \in V$, via:

$$\pi(x)v = x \cdot v.$$  

and vice versa.

Example: Any Lie algebra g can be made into a g-module under the adjoint action, i.e.,

$$x \cdot y = [x, y]$$

Example: If V and V’ are both g-modules, we may make $V \otimes V'$ into a g-module via the action:

$$x \cdot (v \otimes v') = (x \cdot v) \otimes v' + v \otimes (x \cdot v').$$

Now, for some other common definitions. For the following, assume g is a Lie algebra, V is a g-module, and that U and W are subspaces of V.

Definition 4.2. U is called a submodule of V iff $g \cdot U \subset U$.

Definition 4.3. U is called a proper submodule iff $U \neq V$ and $U \neq \{0\}$.

Definition 4.4. V is called irreducible iff V has no proper submodules.

Definition 4.5. V is called completely reducible iff V can be written as a direct sum of submodules W and U, i.e.

$$V = U \oplus W.$$ 

Definition 4.6. V is called indecomposable iff V cannot be written as a direct sum of two proper submodules.
5 The Universal Enveloping Algebra

We now want to discuss an associative algebra that can be constructed from a given Lie algebra. The modules of this associative algebra are in bijective correspondence with the modules of the Lie algebra upon which it is built, but this new algebra is necessarily infinite dimensional. It is called the universal enveloping algebra.

Let $L$ be a Lie algebra.

**Definition 5.1.** We define a Tensor Algebra as follows: Let:

$$T^0 = \mathbb{C}$$
$$T^1 = L$$
$$T^n = L \otimes ... \otimes L \ (n \text{ times}).$$

$$T(L) = \bigoplus_{n \geq 0} T^n$$

We call $T(L)$ the tensor algebra of $L$.

Now, we take the subspace of $T(L)$ generated by the elements

$$x \otimes y - y \otimes x - [x, y] \quad (5.1)$$

and call it $Q(L)$.

**Definition 5.2.** We define the Universal Enveloping Algebra of $L$, denoted $U(L)$, as follows:

$$U(L) = T(L)/Q(L). \quad (5.2)$$

That is, we quotient the tensor algebra of $L$ by $Q(L)$. $U(L)$ is naturally an $L$-module.

6 An Induced Module

Now, suppose we have a vector space $V$ that is only a module for some subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$. We need a way to construct a certain $\mathfrak{g}$-module from $V$. We do this as follows: Consider the vector space

$$U(\mathfrak{g}) \otimes V. \quad (6.1)$$

We know that $V$ is an $\mathfrak{h}$-module, and that $U(\mathfrak{g})$ is a $\mathfrak{g}$-module. Let $I$ be the subspace of $U(\mathfrak{g}) \otimes V$ generated by the following elements: For $h \in \mathfrak{h}$,

$$uh \otimes v - u \otimes h \cdot v$$

and consider the quotient $(U(\mathfrak{g}) \otimes V)/I$. This is a $\mathfrak{g}$-module, and is called an induced module. We denote it as $U(\mathfrak{g}) \otimes_{\mathfrak{h}} V$. We still denote the elements of $U(\mathfrak{g}) \otimes_{\mathfrak{h}} V$ as $u \otimes v$. The action of $\mathfrak{g}$ is given by $g \cdot (u \otimes v) = g \cdot u \otimes v$.

Intuitively, we let elements of $\mathfrak{h}$ pass through the tensor product, but do not allow other elements of $\mathfrak{g}$ to pass through.
7 Further Comments

Please email me any comments about ambiguities, errors/typos (I’m sure there are quite a few), and if you have any other comments. I can be reached at csadowski@gmail.com.