Examples

Here we show by example the existence of $n=2$, $k=3$ polynomials with the following numbers of solutions:

We have at least one example for all of the numbers 0-20 and also for infinitely many solutions!

No Solutions:

\[
X^2 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0
\]

One Solution - One Nondiagonalizable:

\[
X^2 + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} X + \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0
\]

One Solution - One Diagonalizable:

\[
X^2 + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0
\]

Two Solutions - Two Diagonalizable:

\[
X^2 + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix} X + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0
\]

Three Solutions - Two Diagonalizable, One Nondiagonalizable:
\[
X^2 + \begin{pmatrix}
-3 & -1 & 0 \\
0 & -5 & 0 \\
0 & 0 & 0
\end{pmatrix} X + \begin{pmatrix}
2 & 3 & 0 \\
0 & 6 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0
\]

Four Solutions: Four Diagonalizable:

\[
X^2 + \begin{pmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0
\]

Five Solutions: Five Diagonalizable:

\[
X^2 + \begin{pmatrix}
-3 & -1 & 0 \\
0 & -5 & 0 \\
0 & 0 & 0
\end{pmatrix} X + \begin{pmatrix}
2 & 3 & 0 \\
0 & 6 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0
\]

Six Solutions: Six Diagonalizable:

\[
X^2 + \begin{pmatrix}
0 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & 3
\end{pmatrix} X + \begin{pmatrix}
0 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix} = 0
\]

Seven Solutions: Seven Diagonalizable: (Credit: Marla)

\[
X^2 + \begin{pmatrix}
0 & -1 & -13/8 \\
0 & 9 & 9 \\
0 & 0 & 3
\end{pmatrix} X + \begin{pmatrix}
0 & 2 & -13/4 \\
0 & -10 & 18 \\
0 & 0 & 2
\end{pmatrix} = 0
\]

Eight Solutions: Eight Diagonalizable:

\[
X^2 + \begin{pmatrix}
-12 & 0 & 0 \\
0 & -15 & 0 \\
0 & 0 & -18
\end{pmatrix} X + \begin{pmatrix}
-5 & 0 & 0 \\
0 & -7 & 0 \\
0 & 0 & -9
\end{pmatrix} = 0
\]
Nine Solutions: Nine Diagonalizable:

\[ X^2 + \begin{pmatrix} -12 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -18 \end{pmatrix} X + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -9 \end{pmatrix} = 0 \]

Ten Solutions: Ten Diagonalizable:

\[ X^2 + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -2/27 \\ 0 & 0 & -3 \end{pmatrix} X + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2/9 \\ 0 & 0 & 0 \end{pmatrix} = 0 \]

Eleven Solutions: Seven Diagonalizable, Four Nondiagonalizable: (Credit: Marla)

\[ X^2 + \begin{pmatrix} 2 & -1 & 0 \\ 8/3 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} X + \begin{pmatrix} 0 & -2 & 0 \\ 0 & -8/3 & 0 \\ 0 & -3 & -1 \end{pmatrix} = 0 \]

Twelve Solutions: Twelve Diagonalizable:

\[ X^2 + \begin{pmatrix} -2 & 0 & 3/2 \\ 0 & -3 & -1/2 \\ 0 & 0 & 2 \end{pmatrix} X + \begin{pmatrix} -3 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 0 \]

Thirteen Solutions: Thirteen Diagonalizable:

\[ X^2 + \begin{pmatrix} -2 & -3/2 & 0 \\ 0 & -2 & 0 \\ -3/2 & -9/4 & 1 \end{pmatrix} X + \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 9/2 & -9/2 & -2 \end{pmatrix} = 0 \]

Fourteen Solutions: Fourteen Diagonalizable:
\begin{align*}
X^2 + \begin{pmatrix} -3 & 4/3 & 0 \\ 0 & -1 & 0 \\ -1/3 & -2/9 & 1 \end{pmatrix} X + \begin{pmatrix} 0 & -8/3 & 0 \\ 0 & -2 & 0 \\ 1 & 4/9 & -2 \end{pmatrix} &= 0
\end{align*}

Fifteen Solutions: Fifteen Diagonalizable:

\begin{align*}
X^2 + \begin{pmatrix} -4 & 3/4 & 0 \\ 0 & -2 & 0 \\ 0 & -3/2 & 1 \end{pmatrix} X + \begin{pmatrix} 3 & -3/2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \end{pmatrix} &= 0
\end{align*}

Sixteen Solutions: Sixteen Diagonalizable:

\begin{align*}
X^2 + \begin{pmatrix} 5 & -6 & 0 \\ 12 & -11 & 0 \\ -192 & 146 & 3 \end{pmatrix} X + \begin{pmatrix} -24 & 12 & 0 \\ -36 & 18 & 0 \\ 576 & -292 & 2 \end{pmatrix} &= 0
\end{align*}

Seventeen Solutions: Seventeen Diagonalizable:

\begin{align*}
\end{align*}

Eighteen Solutions: Eigenteen Diagonalizable:

\begin{align*}
X^2 + \begin{pmatrix} 9/11 & -8/11 & -7/11 \\ -6/11 & -13/11 & 1/11 \\ 9/11 & 3/11 & -29/11 \end{pmatrix} X + \begin{pmatrix} -26/11 & 16/11 & 21/11 \\ -12/11 & -18/11 & -3/11 \\ 18/11 & -6/11 & -12/11 \end{pmatrix} &= 0
\end{align*}

Nineteen Solutions: Nineteen Diagonalizable:

\begin{align*}
X^2 + \begin{pmatrix} -21/23 & -14/23 & 50/23 \\ -148/23 & 47/23 & -112/23 \\ -128/23 & 22/23 & -95/23 \end{pmatrix} X + \begin{pmatrix} -144/23 & -28/23 & 50/23 \\ 444/23 & 2/23 & -112/23 \\ 384/23 & 44/23 & -118/23 \end{pmatrix} &= 0
\end{align*}
Twenty Solutions: Twenty Diagonalizable:

\[
X^2 + \begin{pmatrix}
\frac{11}{91} & -\frac{82}{91} & \frac{22}{91} \\
\frac{324}{91} & -\frac{41}{91} & \frac{80}{91} \\
\frac{288}{91} & \frac{186}{91} & -\frac{243}{91}
\end{pmatrix}
X + \begin{pmatrix}
-\frac{342}{91} & \frac{246}{91} & -\frac{440}{91} \\
\frac{648}{91} & -\frac{696}{91} & \frac{160}{91} \\
\frac{576}{91} & -\frac{558}{91} & \frac{122}{91}
\end{pmatrix} = 0
\]

Infinite Solutions: Infinite Diagonalizable

\[
X^2 + \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -4
\end{pmatrix} = 0
\]

Infinite Solutions: Infinite Nilpotent

\[
X^2 + \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
X + \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0
\]

Infinite Solutions: Infinite Nilpotent

\[
X^2 + \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0
\]

More to come later! (hopefully!)