

6-critical graphs in the Klein bottle

Daniel Král'

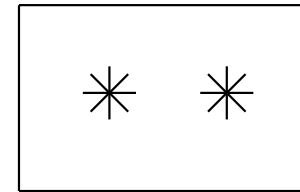
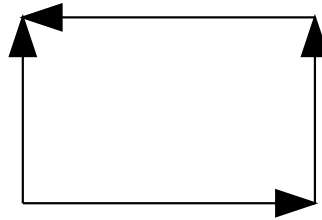
Jan Kynčl

Bernard Lidický

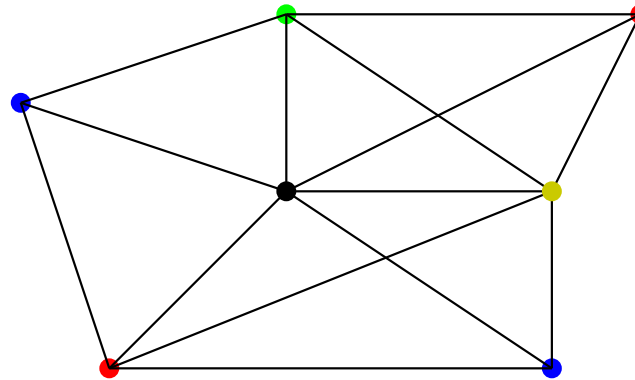
mentors:

Eric Allender, Michael Saks, Van Vu

- **Klein bottle:**



- **graph embedding:** drawing without crossings



- **5-colorable graph:**

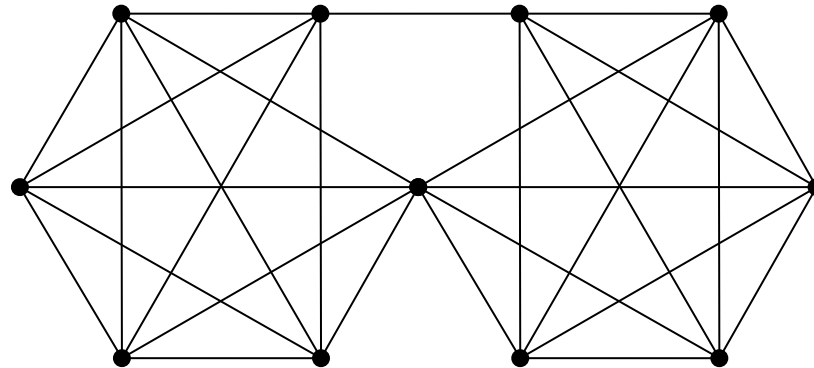
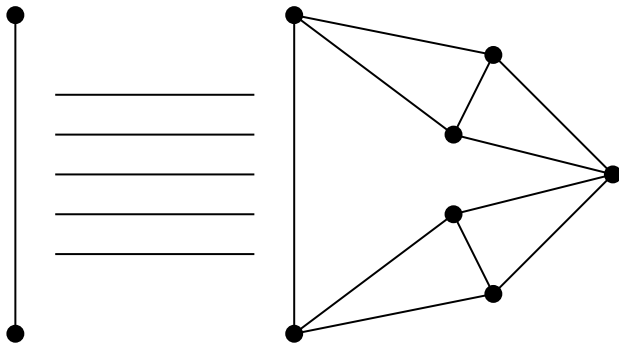
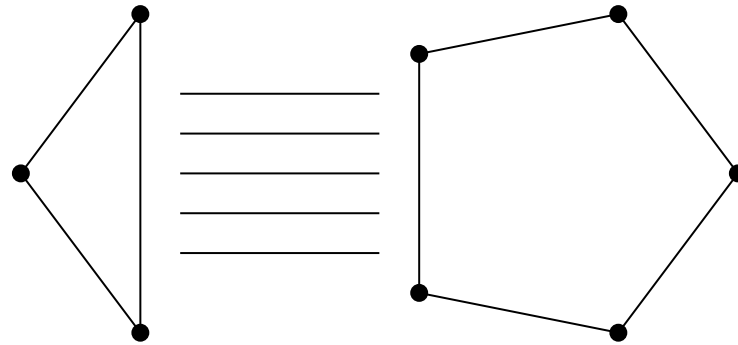
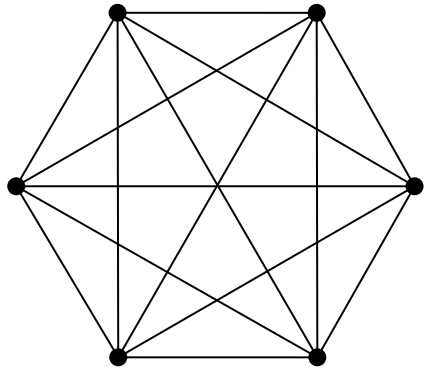
- **6-critical graph:** graph which is not 5-colorable, but each of its proper subgraphs is 5-colorable

Fact: Every graph embedded in the Klein Bottle is 6-colorable.

Theorem: [Thomassen] The set of 6-critical graphs embeddable in the Klein bottle is finite.

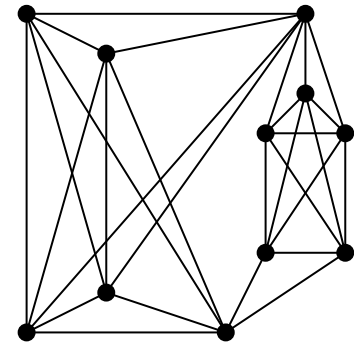
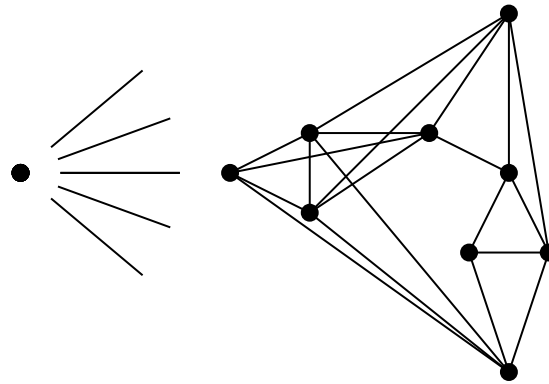
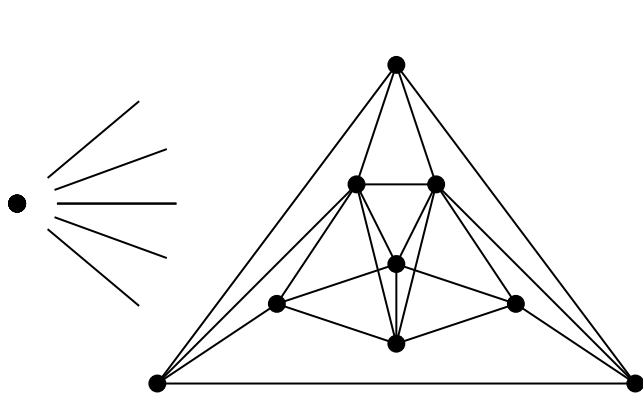
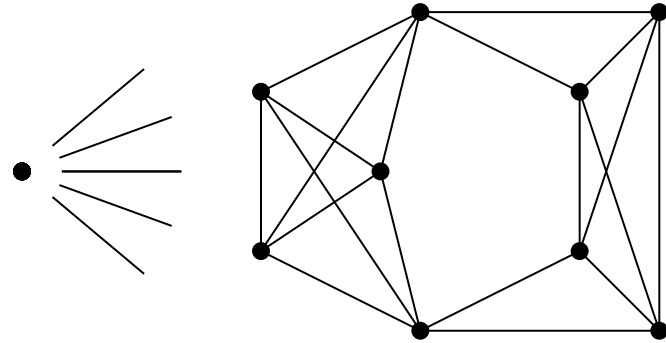
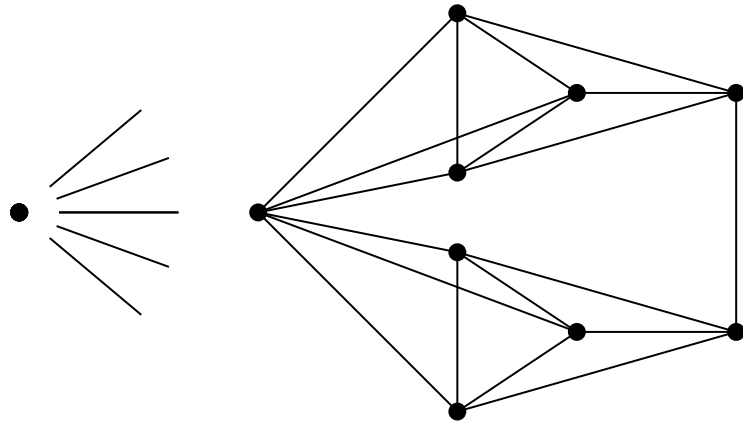
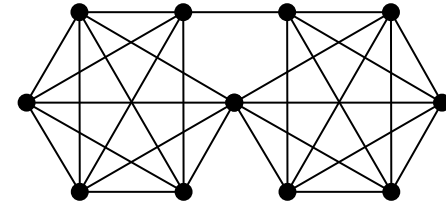
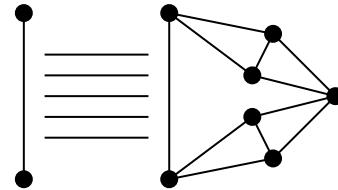
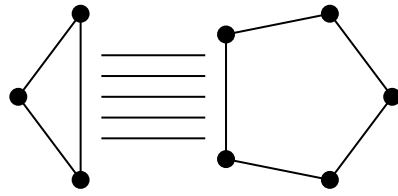
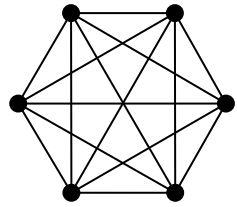
Corollary: There exists a polynomial algorithm for determining whether a given graph embedded in the Klein bottle is 5-colorable.

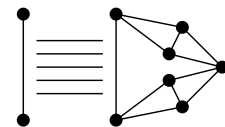
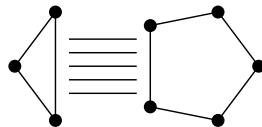
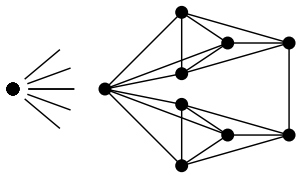
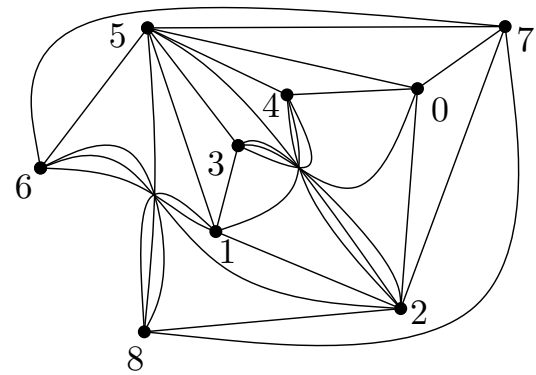
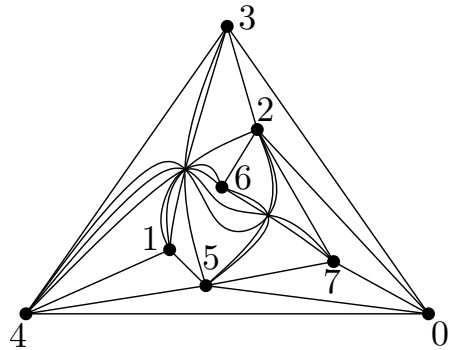
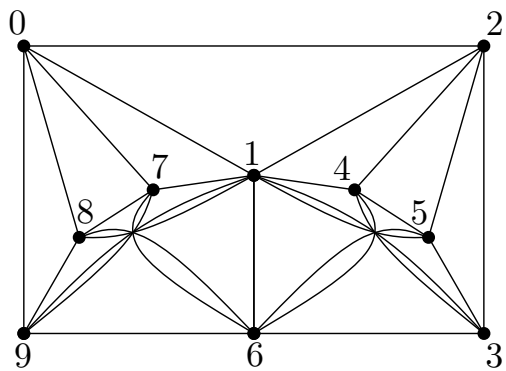
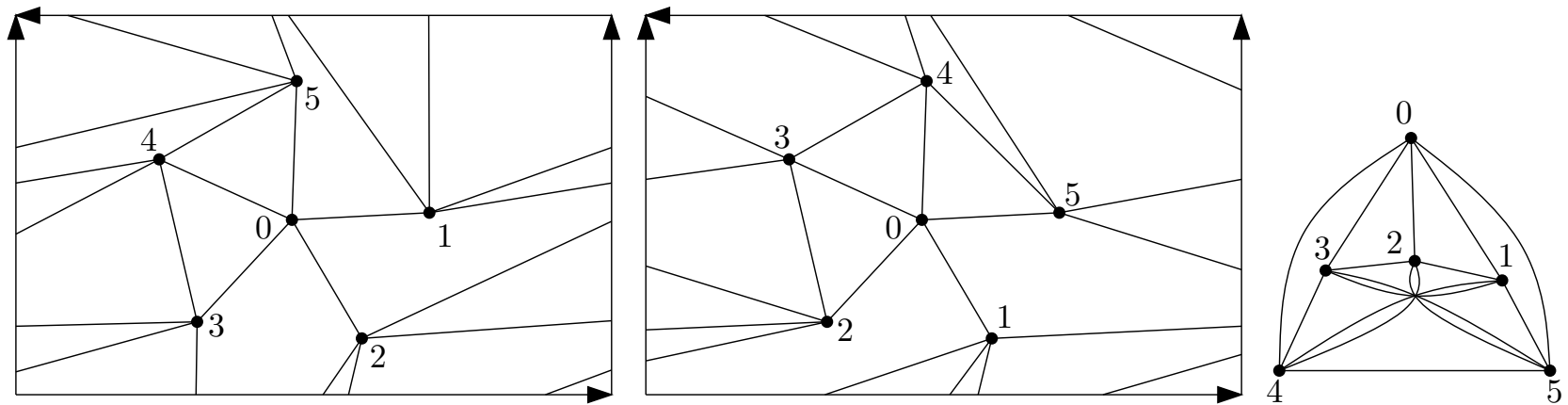
Conjecture: [Thomassen] A graph embedded in the Klein bottle is 5-colorable if and only if it does not contain any of the following 4 graphs as a subgraph:



Our result

Theorem: A graph embedded in the Klein bottle is 5-colorable if and only if it does not contain any of the following 9 graphs as a subgraph:





Outline of the proof

Fact: Each graph embeddable in the Klein bottle is either 6-regular or has a vertex of degree at most 5.

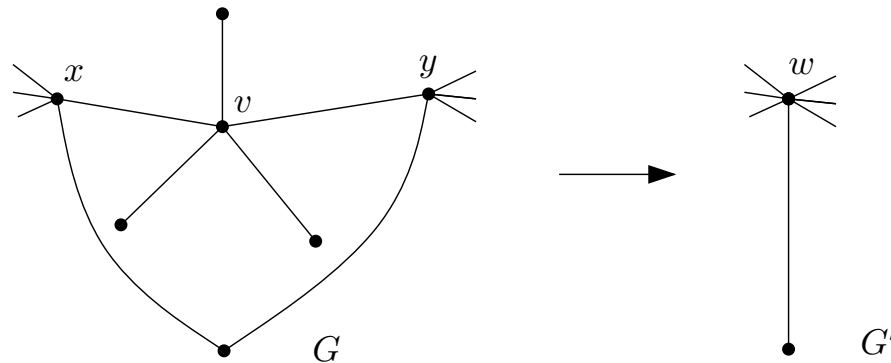
Let G be a graph embedded in the Klein bottle.

- If G is 6-regular, then G is 5-colorable.
- If G has a vertex of degree at most 4, then G is not 6-critical.

\Rightarrow each 6-critical G has a vertex of degree 5.

Contracting operation:

- Let G be a 6-critical graph embedded in the Klein bottle, $G \neq K_6$.
- Let v be a vertex of degree 5.
- v has neighbours x and y which are not connected by an edge.



Observation: G' is not 5-colorable.

$\Rightarrow G'$ has a 6-critical subgraph.

Generating of the 6-critical graphs

- start from K_6
 - generate new graphs using a **decontracting operation**
 - test them for 6-criticality
- two independent computer programs
 - running time: ~ 15 hours
 - debugging time: we hope it's over