Ultimate independence ratios of wheels

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Problem: Find the ultimate independence ratio of odd wheels.

Let us begin with a few definitions. The *independence ratio* of a graph $G = (V, E)$ is the fraction

$$i(G) = \frac{\alpha(G)}{|V(G)|}$$

with $\alpha(G)$ being the maximum size of an independent set of G. Next we define *(square) product* $G \square H$ of graphs G and H by putting

$$V(G \square H) = \{(g, h); g \in V(G), h \in V(H)\} \text{ and}$$

$$E(G \square H) = \{\{(g, h), (g', h')\}; (g, g') \in E(G) \text{ and } h = h', \text{ or } g = g' \text{ and } (h, h') \in E(H)\}.$$

The k-th power of G is defined in the obvious way, as a product of k copies of G:

$$G^k = G \square G \square \ldots \square G.$$

Now we can define the *ultimate independence ratio* of G

$$I(G) = \lim_{k \to \infty} i(G^k).$$

For certain graphs (e.g. Petersen, cycles, perfect graphs) this number is known. Our goal is to find the ultimate independence ratio for *odd wheels.* A *wheel* W_n is a cycle C_n with an additional vertex adjacent to all its vertices.

It is easy to check that for all $k \in \mathbb{N}$

$$i(G^k) \geq \frac{1}{\chi(G^k)} = \frac{1}{\chi(G)} \quad \text{and also} \quad i(G^k) \geq i(G^{k+1})$$

thus the limit exists and the following holds:

$$I(G) \geq \frac{1}{\chi(G)}.$$

Conjecture: $I(W_{2m+1}) = 1/\chi(W_{2m+1}) = 1/4$.

An upper bound is also known. If $\chi_f(G)$ is the fractional chromatic number of G (the minimum total weight that can be distributed on independent sets of G so that the sum of the weights of independent sets covering a vertex is at least 1 for every vertex of G), then

$$I(G) \leq \frac{1}{\chi_f(G)}.$$

This explains why we concentrate on *odd* wheels: for even wheels we have

$$\frac{1}{3} \leq I(W_{2m}) \leq \frac{1}{3}$$

and our problem has a trivial solution.

We first try to establish the ultimate independence ratio for W_5. It can be shown by methods of linear programming that $\chi_f(W_5) = 3.5$ and using the above bounds we get

$$\frac{1}{4} \leq I(W_5) \leq \frac{10}{35}.$$

Not so simple (but also possible by linear programming methods) is to compute $\chi_f(W_5^2)$, which gives us even better upper bound:

$$\frac{1}{4} \leq I(W_5) \leq \frac{11}{41}.$$

It seems that finding the correct ultimate independence ratio for odd wheels (or even for W_5 only) is very difficult and too ambitious for us. Our goal is to lower the upper bound; if we succeed in doing that, we will be satisfied.