ADJOINT FUNCTORS

Glen M. Wilson

glenmatthewwilson@gmail.com

0. INTRODUCTION

The concept of adjoint functors was first defined by Daniel Kan in 1956 [K-1958], and since then, it has proven to be quite useful. The history leading up to Kan’s breakthrough is concisely described in [ML-1971, p. 103].

The motivation for the study of adjoint functors in this paper arose out of the importance of universal constructions such as products, coproducts, free objects, etc. which pervade all of mathematics in a fundamental way. A characterization of universal constructions in terms of adjoint functors and an associated morphism (the unit of the adjunction) is presented, following [H-1970]. However, we present examples of universal construction which, despite their importance and usefulness, seem to lack sufficient “universality”. A section of this paper is focused on developing a stronger definition of universal construction—one which we claim is more “universal”. This definition also gives us a good tool to understanding when a given functor cannot have a left or right adjoint.

Another focus of this paper is to demonstrate the limitations of adjoint functors to explain certain phenomena in mathematics. Our major example is that of the de Rahm cohomology functor $H^*_{dR}$. Conversely, we also demonstrate how the categorical perspective on certain problems can lead to elegant methods which would be considerably more difficult to describe without the categorical language. In particular, we explain a categorical approach to determining if certain universal objects exist in a category.

I would like to thank Dr. Alves for mentoring me during this semester, and Dr. Curtis, Dr. Clifford and Dr. Hingston for helpful discussions about the topology and analysis going on in the last sections. I am also indebted to the wonderful website mathoverflow.net for the useful discussions when I was getting lost in the abstraction. I would especially like to thank Andrew Stacey, Reid Barton and Chris Schommer-Pries.

1. Basic Notions of Category Theory

In this section, we develop the some important categorical definitions and ideas which will be used throughout this paper. For a more complete treatment, the interested reader should consult either [ML-1971], [H-1970] or [M-1967].

Definition 1.1: A metacategory (which we typically denote as $\mathcal{C}$ or $\mathcal{D}$) is a pair $\mathcal{C} = (\mathbf{O}_\mathcal{C}, \mathcal{M}_\mathcal{C})$ where $\mathbf{O}_\mathcal{C}$ is considered to be the collection of objects of $\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$ is considered a collection of morphisms (or arrows) between the objects of $\mathcal{C}$ that are subject to a few rules.

1. For a morphism $f$ of $\mathcal{C}$, there are two associated objects: $\text{dom}(f)$ and $\text{cod}(f)$. If $\text{dom}(f) = X$ and $\text{cod}(f) = Y$, we will typically depict this in a diagram as:

$$f : X \longrightarrow Y \quad \text{or} \quad X \xrightarrow{f} Y$$

2. For any object $X$ of $\mathcal{C}$, there is a morphism $\text{id}_X : X \rightarrow X$ which makes the following diagram commute for any $f, g \in \mathcal{M}_\mathcal{C}$ with $\text{cod}(f) = \text{dom}(g) = X$.
3. Every composable pair of morphisms \((f, g)\)—that is, \(\text{cod}(f) = \text{dom}(g)\)—has the composition \(f \circ g \in M_C\).

4. If \((f, g)\) and \((g, h)\) are composable pairs of morphisms, then \((f \circ g) \circ h = f \circ (g \circ h)\).

**Definition 1.2:** A category \(\mathcal{C}\) is a metacategory in which \(\mathcal{O}_C, M_C\) are classes and for all \(X, Y \in \mathcal{C}\) the morphism sets \(\mathcal{C}(X, Y)\) are sets.

**Remark 1.3:** We use this definition of category in order to speak about adjunctions in a straightforward manner. The above definitions follow [ML-1971, 7-10], [M-1967] and [A-2004]. A more general approach is to define “category” to be what we define a “metacategory” to be. There are obvious benefits to the more general viewpoint, but it complicates our discussion of adjoints. We thus recommend the interested reader to [http://ncatlab.net](http://ncatlab.net) for more details on the abstract approach.

**Definition 1.4:** A small category is a category \(\mathcal{C}\) for which \(\mathcal{O}_C\) and \(M_C\) are sets. In general, by appending “small” to a noun, we imply that we restrict our attention to sets. In this case, the relevant objects are \(\mathcal{O}_C\) and \(M_C\) which we require to be small.

**Notation 1.5:** Let \(\mathcal{C}\) be a category. For objects \(X, X' \in \mathcal{C}\), we denote the collection of all morphisms \(f \in \mathcal{C}\) with \(\text{dom}(f) = X\) and \(\text{cod}(f) = X'\) by \(\mathcal{C}(X, X')\).

**Definition 1.6:** Let \(\mathcal{C}\) and \(\mathcal{D}\) be categories. A functor is a transformation \(F : \mathcal{C} \to \mathcal{D}\) which assigns to each object \(X \in \mathcal{C}\) an object \(FX \in \mathcal{D}\), to each morphism \(f \in \mathcal{C}(X, X')\) a morphism \(Ff \in \mathcal{D}(FX, FX')\) and satisfies the following properties:

1. \(F(\text{id}_X) = \text{id}_{FX}\);
2. \(F(f \circ g) = Ff \circ Fg\).

**Example 1.7:**

1. Define \(\text{Cat}\) to be the category with objects all small categories and morphisms all functors between small categories.

2. Take \(\text{Ab}\) to be the category of Abelian groups with morphisms being group homomorphisms.

3. Define \(\text{Gp}\) to be the category of groups with group homomorphisms.

4. Define \(\text{Top}\) to be the category of topological spaces with morphisms continuous maps.

5. Define \(\text{Haus}\) to be the category of Hausdorff topological spaces with continuous maps.

6. Define \(\text{Man}\) to be the category of those Hausdorff spaces which are locally Euclidean and paracompact with morphisms continuous functions.

7. Define \(\text{SmMan}\) to be the category of smooth manifolds with smooth functions.

8. Define \(\text{AMan}\) to be the category of analytic real manifolds with analytic functions.
Definition 1.8: A full functor is a functor \( F : \mathcal{C} \to \mathcal{D} \) which is surjective on morphism sets. That is, for any \( f \in \mathcal{D}(FX, FX') \) there exists \( g \in \mathcal{C}(X, X') \) such that \( f = Fg \).

A faithful functor, or an embedding is a functor \( F : \mathcal{C} \to \mathcal{D} \) which is injective on morphism sets, that is for \( f, f' \in \mathcal{C}(X, X') \), if \( Tf = Tf' \), then \( f = f' \). Note that a functor can be faithful without being injective on objects.

Definition 1.9: Let \( F, G : \mathcal{C} \to \mathcal{D} \) be functors. A natural transformation \( \tau : F \to G \) is a family of morphisms \( \tau_X : FX \to GX \) which make for any \( f \in \mathcal{C}(X, X') \) the following diagram commute:

\[
\begin{array}{ccc}
FX & \xrightarrow{\tau_X} & GX \\
\downarrow Ff & & \downarrow Gf \\
FX' & \xrightarrow{\tau_{X'}} & GX'
\end{array}
\]

A natural equivalence is a natural transformation in which \( \tau_X \) is an invertible morphism in \( \mathcal{D} \) for all \( X \in \mathcal{C} \). We write \( \tau : F \sim G \) to denote a natural equivalence.

Definition 1.10: An isomorphism of categories is a full, faithful functor which is also a bijection on objects. A weaker notion is that of equivalence.

An equivalence of categories is a functor \( F : \mathcal{C} \to \mathcal{D} \) for which there is a functor \( G : \mathcal{D} \to \mathcal{C} \) and natural equivalences \( F \circ G \sim \text{id}_\mathcal{D} \) and \( G \circ F \sim \text{id}_\mathcal{C} \).

Definition 1.11: A concrete category is a category \( \mathcal{C} \) equipped with a faithful functor \( U : \mathcal{C} \to \text{Set} \). As we can may identify a morphism \( f \in \mathcal{M}\mathcal{C} \) with \( Uf \), we may think of each object of \( \mathcal{C} \) as having an “underlying” set and each morphism \( f \) as a set function \( Uf \).

Definition 1.12: Given a category \( \mathcal{C} \), the opposite category of \( \mathcal{C} \), denoted \( \mathcal{C}^{\text{op}} \) is the category with \( \mathcal{C}^{\text{op}} = \mathcal{O}\mathcal{C} \) and \( \mathcal{C}^{\text{op}}(X, X') = \mathcal{C}(X', X) \).

Another way to see this is to define for a morphism \( f \in \mathcal{M}\mathcal{C} \) the opposite morphism of \( f \) to be a “symbolic” arrow \( f^{\text{op}} : \text{cod}(f) \to \text{dom}(f) \). Then \( \mathcal{M}\mathcal{C}^{\text{op}} = \{ f^{\text{op}} \mid f \in \mathcal{M}\mathcal{C} \} \). The opposite morphisms are defined so that for a composable pair \( f, g \), \( (f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}} \).

In essence, all we are doing is drawing all of the arrows in \( \mathcal{C} \) backwards.

Remark 1.13: The opposite category of a concrete category (like \( \text{Gp} \) or \( \text{Set} \)) is not always concrete.

Definition 1.14: A contravariant functor \( F : \mathcal{C} \to \mathcal{D} \) is a functor \( \mathcal{C}^{\text{op}} \to \mathcal{D} \). We make the convention that the symbol \( F \) of a contravariant functor always represents the functor \( F : \mathcal{C}^{\text{op}} \to \mathcal{D} \).

Definition 1.15: Given a functor \( F : \mathcal{C} \to \mathcal{D} \), the opposite functor \( F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}} \) is defined by \( F^{\text{op}}(f^{\text{op}} : X' \to X) = (Ff)^{\text{op}} : FX' \to FX \).

Definition 1.16: An indexing category is a small category \( I \), where \( \mathcal{M}I = \{ \text{id}_x \mid x \in \mathcal{O}I \} \).

Definition 1.17: For a small category \( I \), we define \( \mathcal{C}^{I} \) to be the category with objects all functors \( F : I \to \mathcal{C} \) and morphisms all natural transformations between functors \( F \in \mathcal{O}\mathcal{C}^I \).

Notation 1.18: In our commutative diagrams, dotted arrows represent unique induced maps or unique maps whose existence is in question, while dashed arrows stand for induced maps or maps whose existence is in question without any conditions on uniqueness. All diagrams are to be assumed commutative unless otherwise noted.

2. Basic Properties of Adjoint Functors
Definition 2.19: Let \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) be functors. We say that \( F \) is left adjoint to \( G \) (or equivalently \( G \) is right adjoint to \( F \)) if there exists a natural equivalence \( \eta : \mathcal{D}(F -, -) \to \mathcal{C}(-, G-) \) between the functors \( \mathcal{D}(F -, -), \mathcal{C}(-, G-) : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Set} \). In this case, we write \( \eta : F \dashv G \).

Given \( \eta : F \dashv G \), we now list the most important properties of such a natural equivalence.

1. As \( \eta \) is a natural equivalence, given \( \alpha : X' \to X \) and \( \beta : Y \to Y' \) the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{D}(FX,Y) & \xrightarrow{\eta_{X,Y}} & \mathcal{C}(X,GY) \\
F\alpha^* & \downarrow & \alpha^* \\
\mathcal{D}(FX',Y) & \xrightarrow{\eta_{X',Y'}} & \mathcal{C}(X',GY)
\end{array}
\]

captures the fact that \( \eta \) is a natural transformation. For \( \phi \in \mathcal{D}(FX,Y) \), the equation \( \eta(\beta \circ \phi \circ F\alpha) = G\beta \circ \eta(\phi) \circ \alpha \) obtained by walking around the perimeter of the above diagram is equivalent to the naturality of \( \eta \). By setting one of \( \beta \) or \( \alpha \) as the identity, we recover the commutativity of the top and bottom squares respectively.

We thus have the fundamental equation

\[ \eta(\beta \circ \phi \circ F\alpha) = G\beta \circ \eta(\phi) \circ \alpha \]

which implies

\[ \eta(\phi \circ F\alpha) = \eta(\phi) \circ \alpha \]
\[ \eta(\beta \circ \phi) = G\beta \circ \eta(\phi). \]

We can interpret these equations as a distributivity property of \( \eta \).

2. Taking \( Y = FX \) and \( \phi = \text{id}_{FX} \), we define the unit of the adjunction to be the natural transformation \( \varepsilon : \text{id}_\mathcal{C} \to GF \) with \( \varepsilon_X := \eta(\text{id}_{FX}) : X \to GFX \).

Now taking \( X = GY \), we define the counit of the adjunction to be the natural transformation \( \delta : FG \to \text{id}_\mathcal{D} \) defined by \( \delta_Y := \eta^{-1}(\text{id}_{GY}) \).

To prove that these are natural transformations, we must show that the following diagram is commutative, i.e. \( \varepsilon_X \circ \alpha = GF\alpha \circ \varepsilon_{X'} \).

\[
\begin{array}{ccc}
X' & \xrightarrow{\varepsilon_{X'}} & GFX' \\
\downarrow{\alpha} & & \downarrow{GF\alpha} \\
X & \xrightarrow{\varepsilon_X} & GFX
\end{array}
\]

One may use the distributivity equations above, or consult the following diagram which is commutative by the naturality of \( \eta \) to determine that \( \eta(F\alpha) = \varepsilon_X \circ \alpha \).
Alternatively, we compute
\[
\eta(id \circ F \alpha) \quad \xrightarrow{\eta(id) \circ \alpha} \quad \varepsilon_X \circ \alpha
\]

Entirely similar computations show that the counit \(\delta\) is a natural transformation as well.

3. The following two equations hold as well:
\[
\delta F \circ F \varepsilon = \text{id}
\]
\[
G \delta \circ \varepsilon G = \text{id}
\]

To prove the validity of these two equations, we compute using the fundamental equation
\[
\eta(\delta_{FX} \circ F \varepsilon_X) = \eta(\delta_{FX}) \circ \varepsilon_X = \varepsilon_X = \eta(\text{id}_{FX})
\]
which implies \(\delta_{FX} \circ F \varepsilon_X = \text{id}_{FX}\) as \(\eta\) is a bijection. A similar computation establishes the other equation.

4. The following two equations explicitly describe the natural equivalence \(\eta\) in terms of the unit, counit and functors \(F\) and \(G\).
\[
\eta(\psi) = G \psi \circ \varepsilon_X, \text{ with } \psi : FX \to Y;
\]
\[
\eta^{-1}(\zeta) = \delta_Y \circ F \zeta, \text{ with } \zeta : X \to GY.
\]

One can prove the first equation with the following diagram, chasing \(\text{id}_{FX}\) around:
The above objects derived from the adjunction $\eta$ are indeed enough to reconstruct it. We formulate this in the following proposition.

**Proposition 2.20:** If $\varepsilon : \text{id} \to GF$ and $\delta : FG \to \text{id}$ are natural transformations and if the equation $\delta F \circ F \varepsilon = \text{id}$ and $G \delta \circ \varepsilon G = \text{id}$ hold, then $\eta : F \dashv G$, defined by $\eta(\phi) = G \phi \circ \varepsilon_X$, is a natural equivalence which shows $F$ is left adjoint to $G$. Furthermore, $\varepsilon$ and $\delta$ are the unit and counit of the adjunction $\eta$ respectively.

Conversely, if $\eta : F \dashv G$ is a natural equivalence, then $\varepsilon_X := \eta(\text{id}_{FX})$ and $\delta_Y := \eta^{-1}(\text{id}_{GY})$ define natural transformations which satisfy the above equations.

**Proof.** The first part of the proposition is the only thing that remains to be proven. We thus show that $\eta$ is natural by verifying that the equation $\eta(\beta \circ \phi \circ F \alpha) = \delta \circ F \eta(\phi) \circ \alpha$ holds:

$$\eta(\beta \circ \phi \circ F \alpha) = G(\beta \circ \phi \circ F \alpha) \circ \varepsilon_X,$$

$$G \beta \circ G \phi \circ GF \circ \varepsilon_X,$$

$$G \beta \circ G \phi \circ \varepsilon_X \circ \alpha \text{ (by naturality of $\varepsilon$)}$$

$$G \beta \circ \eta(\phi) \circ \alpha.$$

We now define $\xi : \mathcal{C}(\cdot, G\cdot) \to \mathcal{D}(F\cdot, \cdot)$ by $\xi(\psi) := \delta_Y \circ F \psi$, for $\psi : X \to GY$. We show that $\xi$ is inverse to $\eta$, thus proving that $\eta$ is a natural equivalence.

$$\xi(\eta(\phi)) = \delta_Y \circ F \circ \eta(\phi)$$

$$= \delta_Y \circ F(G \phi \circ \varepsilon_X)$$

$$= \delta_Y \circ FG \phi \circ F \varepsilon_X$$

$$= \phi \circ \delta_{FX} \circ F \varepsilon_X, \text{ (by naturality of $\delta$)}$$

$$= \phi.$$

One similarly shows that $\eta \xi = \text{id}$, from which we conclude $\xi = \eta^{-1}$ and $\eta$ is a natural equivalence. Thus $\eta : F \dashv G$. Define $\varepsilon'_X := \eta(\text{id}_{FX})$ and $\delta'_Y := \eta^{-1}(\text{id}_{GY})$. We compute

$$\varepsilon'_X = \eta(\text{id}_{FX})$$

$$= G(\text{id}_{FX}) \circ \varepsilon_X$$

$$= \text{id}_{GF X} \circ \varepsilon_X$$

$$= \varepsilon_X,$$

and

$$\delta'_Y = \eta^{-1}(\text{id}_{GY})$$

$$= \delta_Y \circ F(\text{id}_{GY})$$

$$= \delta_Y \circ \text{id}_{FGY}$$

$$= \delta_Y,$$

as desired. ▲

**Proposition 2.21:** If $\eta : F \dashv G$ and $\eta' : F \dashv G'$, then there exists a natural equivalence between $G$ and $G'$. We remark that for all $Y \in \mathcal{D}$, we have $GY \cong G'Y$. Alternatively, $G$ determines $F$ up to natural equivalence.

**Proof.** We prove the first claim. We have the natural equivalence
We thus define \( \theta_Y := \eta' \circ \eta^{-1}(\text{id}_{GY}) : GY \to G'Y \), i.e. \( \theta = G'\delta \circ \varepsilon'G \). It follows from the above derived equations that \( \tilde{\theta} := G\delta \circ \varepsilon G' \) is the inverse to \( \theta \). Thus \( \theta \) induces an equivalence between \( G \) and \( G' \) as desired. ▲

**Proposition 2.22:** Let \( F : \mathcal{C} \to \mathcal{D}, F' : \mathcal{D} \to \mathcal{C} \) be functors, and suppose there exist \( G \) and \( G' \) such that \( \eta : F \dashv G \) and \( \eta' : F' \dashv G' \). Then \( \eta_{-,-} \circ \eta_{F,-} : F'F \dashv GG' \).

**Proof.** Clear. ▲

### 3. (Co-)Universal Constructions

We now have the necessary terminology and properties to define and study universal constructions in mathematics. The following definitions follow essentially from [H-1970].

**Definition 3.23:** Let \( \mathcal{C}, \mathcal{D} \) be categories. A universal construction with respect to the functor \( G : \mathcal{D} \to \mathcal{C} \) is a left adjoint to \( G \) with the unit of the adjunction.

**Definition 3.24:** Let \( \mathcal{C}, \mathcal{D} \) be categories. A couniversal construction with respect to the functor \( F : \mathcal{C} \to \mathcal{D} \) is a right adjoint to \( F \) with the counit of the adjunction.

In the literature, the term “universal construction” is often abused and used for both universal and couniversal constructions. We use the terminology precisely as defined above. In addition, we also use the term “(co-)universal property” to mean the property that a (co-)universal construction determines in the categories \( \mathcal{C} \) and \( \mathcal{D} \). We will discuss this terminology more in §4.

We can now delve into specific examples of universal and couniversal constructions. As we will see, some (co-)universal constructions can only be talked about in a specific category. It is therefore a main point of this paper to define and investigate (co-)universal constructions which one can talk about in any category. It then becomes clear with the very simple result §3 that a functor \( F : \mathcal{C} \to \mathcal{D} \) that has an adjoint preserves important information about the category.

**Example 3.25:** Let \( \mathcal{C} \) be a category; let \( I \) be an indexing category and consider the diagonal functor \( P : \mathcal{C} \to \mathcal{C}^I \), i.e. \( P(X)(\cdot) = X \) and \( P(X)(\cdot \to \cdot) = \text{id}_X \). A right adjoint \( R \) to \( P \) along with the counit \( \delta \) of the adjunction \( \eta : P \dashv R \) defines the product in the category \( \mathcal{C} \) over the indexing category \( I \). We often denote \( R \) by \( \prod \). This is a couniversal construction, yet contrary to our intuition, it is called the product (and not the coproduct).

**Proposition 3.26:** The above description of the product is equivalent to the following “usual” universal property: the object \( R\{X_i\} \) with morphisms \( \pi_i : R\{X_i\} \to X_i \) is the product of \( \{X_i\} \) in \( \mathcal{C} \) if for any other object \( Y \in \mathcal{C} \) with morphisms \( \phi_i : Y \to X_i \) there exists a unique morphism \( \phi' : Y \to R\{X_i\} \) such that \( \pi_i \circ \phi' = \phi_i \) for all \( i \in I \).

**Proof.** Given \( \eta : P \dashv R \), and \( \{X_i\} \in \mathcal{C}^I \), we show that \( R\{X_i\} \) along with \( \delta \) satisfies the “usual” universal property. That is, we assume we are given a diagram
in $\mathcal{C}'$ with $Y$ and $\phi$ arbitrary, and seek a unique morphism $\phi'$ which makes the diagram commute. As $\eta : \mathcal{C}'(PY, \{X_i\}) \rightarrow \mathcal{C}(Y, R\{X_i\})$, we take $\phi' := \eta(\phi) : Y \rightarrow R\{X_i\}$ which satisfies $\delta \circ P\phi' = \eta^{-1}(\phi') = \eta^{-1}(\eta(\phi)) = \phi$, i.e. the above diagram can be completed in a unique way so that it commutes.

We now prove the other direction. We assume that for any $\{X_i\} \in \mathcal{O}\mathcal{C}'$ there is a morphism $\delta : PR\{X_i\} \rightarrow \{X_i\}$ that the following holds: for any $Y \in \mathcal{C}$ with a morphism $\phi : PY \rightarrow \{X_i\}$ there exists a unique morphism $\phi' : Y \rightarrow R\{X_i\}$ such that $\phi = \delta \circ \phi'$.

We begin by showing that $R : \mathcal{C}' \rightarrow \mathcal{C}$ is a functor. We thus suppose we have a diagram

\[
\begin{array}{ccc}
PR\{Z_i\} & \xrightarrow{\beta} & \{Z_i\} \\
\downarrow{\eta} & & \downarrow{\phi} \\
PR\{Y_i\} & \xrightarrow{\alpha} & \{Y_i\} \\
\downarrow{\alpha} & & \downarrow{\phi'} \\
PR\{X_i\} & \xrightarrow{\gamma} & \{X_i\}
\end{array}
\]

and seek to define for a morphism $\alpha : \{X_i\} \rightarrow \{Y_i\} \in \mathcal{C}$ an induced morphism $R\alpha : R\{X_i\} \rightarrow R\{Y_i\}$ and show that $R(\beta \circ \alpha) = R\beta \circ R\alpha$. We define $R\alpha := \alpha'$. We see that the above diagram induces a commutative diagram

\[
\begin{array}{ccc}
PR\{Z_i\} & \xrightarrow{\beta} & \{Z_i\} \\
\downarrow{PR(\beta\eta)} & & \downarrow{\beta\phi'} \\
PR\{Y_i\} & \xrightarrow{\alpha} & \{Y_i\} \\
\downarrow{PR(\alpha\phi')} & & \downarrow{\beta\alpha\phi'} \\
PR\{X_i\} & \xrightarrow{\gamma} & \{X_i\}
\end{array}
\]

We remark that the equality $PR\beta\eta \circ PR\alpha\phi' = PR(\beta\alpha\phi')$ follows by the uniqueness of the induced maps from our assumption. We thus have concluded that $R : \mathcal{C}' \rightarrow \mathcal{C}$ is a functor.

We now show the morphism $\delta : PR\{X_i\} \rightarrow \{X_i\}$ defines a natural transformation $\delta : PR \rightarrow \text{id}_{\mathcal{C}'}$. Let $\alpha : \{X_i\} \rightarrow \{Y_i\} \in \mathcal{C}'$. Then by our assumption, we obtain the commutative diagram

\[
\begin{array}{ccc}
PR\{Z_i\} & \xrightarrow{\beta} & \{Z_i\} \\
\downarrow{PR(\beta\eta)} & & \downarrow{\beta\phi'} \\
PR\{Y_i\} & \xrightarrow{\alpha} & \{Y_i\} \\
\downarrow{PR(\alpha\phi')} & & \downarrow{\beta\alpha\phi'} \\
PR\{X_i\} & \xrightarrow{\gamma} & \{X_i\}
\end{array}
\]
which proves that $\delta$ is a natural equivalence. Thus we may define a natural transformation $\xi : \mathcal{C}(Y,R\{X_i\}) \to \mathcal{C}(PY,\{X_i\})$ by $\xi \psi := \delta \circ P\psi$. To complete the proof, we now show $\xi$ is a natural equivalence. Define $\eta : \mathcal{C}(PY,\{X_i\}) \to \mathcal{C}(Y,R\{X_i\})$ by setting $\eta(\phi) := \phi'$, i.e. the induced map $\phi' : Y \to R\{X_i\}$ seen in our assumption. We verify that $\eta \xi = \text{id} = \xi \eta$. Because

we have

$$\eta(\xi \phi) = \eta(\delta \circ P\phi)$$

$$= \phi.$$  

Similarly, we consider $\psi : PY \to \{X_i\}$ and obtain

$$\xi(\eta(\psi)) = \delta \circ P\eta(\psi)$$

$$= \psi$$  

from the commutative diagram

We thus conclude that $\xi = \eta^{-1}$, and hence $\eta : P \rightleftarrows R$ as desired.

Many familiar categories admit small products and coproducts, and the constructions are what one expects. An interesting example of products and coproducts arises when we consider the case of a partially ordered set $(X, \leq)$ which we interpret as a category with $OX = X$ and $MX = \{x \to y : x \leq y\}$. The product in such a category (if it exists) is easily seen to be the infimum with the help of the following diagram:
Interpreting the above diagram yields the property: for any $Y \in X$ such that $Y \leq r_i$ for all $i \in I$, we then have $Y \leq \prod \{r_i\}$, which is the defining condition for the infimum.

**Example 3.27:** Let $\mathcal{C}$ be a category; let $I$ be an indexing set (i.e. a small category with only the identity morphisms) and consider the diagonal functor $P : \mathcal{C} \to \mathcal{C}^I$. A left adjoint $L$ to $P$ along with the unit $\varepsilon$ of the adjunction $\eta : L \dashv P$ defines the coproduct in the category $\mathcal{C}$ over the indexing set $I$. This is a universal construction.

**Example 3.28:** Let $\mathcal{C}$ be a concrete category. Consider the faithful functor $U : \mathcal{C} \to \mathsf{Set}$—typically called the underlying functor—which comes with the assumption that $\mathcal{C}$ is concrete. The universal construction with respect to $U$ defines the universal property of free objects. That is, a left adjoint $L \dashv U$ along with the unit $\varepsilon$ of the adjunction.

One similarly defines cofree objects as the couniversal construction with respect to $U$.

**Example 3.29:** The limit (some call this the inverse limit) is similar in construction to the product. The functor of interest is once again the diagonal functor $\lim$ referred to as the limit!). We typically denote the limit with respect to a functor $P$ then yields the limit, while the universal construction yields the colimit (also frequently referred to as the limit!). We typically denote the limit with respect to a functor $F \in \mathcal{C}^I$ as $\lim F$ or $\lim F$, while the colimit is denoted by either $\mathrm{colim} F$ or $\lim F$.

The classical example of a limit which is not a product is found in $\mathsf{Rng}$ when $I = (\mathbb{N}; \geq)$, i.e. the category generated by

\[
\begin{array}{ccccccc}
\cdots & 4 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 1.
\end{array}
\]

For some functor $X : I \to \mathsf{Rng}$, a description of the product of the $X_i$ is given by

\[
\prod_i X_i = \{ x : \mathbb{N} \to \bigcup_i X_i \mid x_n := x(n) \in X_n \}
\]

with the usual component-wise operations. Then the limit of a functor $X : I \to \mathsf{Rng}$ is given by

\[
\lim X = \{ x \in \prod_i X_i \mid x_{ij} = x_j, \forall i \geq j \}
\]

where $X_{ij} = X(f_{ij})$.

Specifically, we consider the functor $\mathcal{R}^p : I \to \mathsf{Rng}$ with $\mathcal{R}^p(i) := \mathbb{Z}_{p^i}$ and $\mathcal{R}^p_i : \mathbb{Z}_{p^i} \to \mathbb{Z}_{p^{i+1}}$ the canonical surjection. The object $\lim \mathcal{R}^p$ is called the $p$-adic integers. The elements of $\lim \mathcal{R}^p$ are often called coherent sequences.

Observe that in the previous example, the limit was a subobject of the product. This is true in general. By duality, a similar relationship is shared between colimits and coproducts.

The definition of monomorphism (and by duality, epimorphism) will be developed in the course of the proof. Of course, [ML-1971] is the standard reference.

**Proposition 3.30:** Let $\mathcal{C}$ be a category which has small products. For a small category $I$, define its underlying indexing category $I'$, i.e. $OI' = OI$ and $MI' = \{ \text{id}_x \mid x \in OI' \}$. Suppose there is a right adjoint to $P_i : \mathcal{C} \to \mathcal{C}^I$. Then for $X \in \mathcal{C}^I$ and $U\mathcal{X} : I' \to \mathcal{C}$ is the induced functor, there exists a monomorphism $\lim X \to \prod U\mathcal{X}$. By duality, there is an epimorphism $\prod U\mathcal{X} \to \colim X$.

**Proof.** Define $U : \mathcal{C}^I \to \mathcal{C}$ by $U(\mathcal{X} : I \to \mathcal{C}) : I' \to \mathcal{C}$, by $(U\mathcal{X})i = \mathcal{X}(i)$ on objects of $i \in OI$. For $\tau : \mathcal{X} \to \mathcal{Y}$ define $U(\tau) : U\mathcal{X} \to U\mathcal{Y}$ by $U(\tau)_i = \tau_i$ for all $i \in I$. This is evidently a functor.

It is clear that the inside triangle of the following diagram commutes.
The counit of the adjunction defining the limit gives us a map \( \delta_X : P\lim X \rightarrow X \in \mathcal{E}^I \). The underlying maps give us \( U\delta_X \in \mathcal{E}^I \). Through the adjunction for the product \( \eta : \mathcal{E}^I(P\lim X, UX) \simeq \mathcal{E}(\lim X, U\prod UX) \) consider \( \overline{\delta} = \eta(\delta_X) \). By the universality of the product, the following diagram commutes

\[
\begin{array}{ccc}
P\Pi X & \xrightarrow{\delta_U} & UX \\
P\lim X & \xrightarrow{\overline{\delta}} & P\lim X
\end{array}
\]

To show that \( \overline{\delta} : \lim X \rightarrow \prod UX \) is a monomorphism, we need to show that for any \( Y \) and any \( f_1, f_2 : Y \rightarrow \lim X \) such that \( \overline{\delta}f_1 = \overline{\delta}f_2 \), it is then the case that \( f_1 = f_2 \).

Through the adjunction \( \mathcal{C}(Y, \lim X) \simeq \mathcal{E}^I(PY, X) \), there correspond unique \( \overline{f}_i : PY \rightarrow X \) such that \( \delta_X \circ Pf_i = \overline{f}_i \) for \( i = 1, 2 \). That is the following diagram commutes for each \( i \)

\[
\begin{array}{ccc}
P\lim X & \xrightarrow{\delta_X} & X \\
Pf_i & & \downarrow \overline{f}_i \\
PY & & \end{array}
\]

Splicing the two diagrams above together, while also applying \( U \) to the one directly above, we obtain

\[
\begin{array}{ccc}
P\Pi U X & \xrightarrow{\delta_U} & UX \\
P\lim X & \xrightarrow{U\overline{f}_i} & U\lim X \\
PY & \xrightarrow{Pf_i} & PY
\end{array}
\]

Note that the individual maps that \( U\overline{f}_i \) is comprised of do not change; the fact that they satisfy certain commutativity properties with the diagram \( X \) is all that is forgotten.

Observe now that as \( \overline{\delta}f_1 = \overline{\delta}f_2 \in \mathcal{C}(Y, U\prod UX) \simeq \mathcal{E}^I(PY, UX) \), the maps \( U\overline{f}_1 = U\overline{f}_2 \). It follows that \( \overline{f}_1 = \overline{f}_2 \), and therefore \( f_1 = f_2 \) as desired. ◼️

We remind the reader that the notions of monomorphism and epimorphism aren’t always what one expects. The typical example is that in Haus and Man a map \( f : X \rightarrow Y \) is an epimorphism if and only if \( \text{im } f \) is a dense subset of \( Y \) by the Coincidence Theorem [S-1992]. In particular, \( \iota : \mathbb{Q} \hookrightarrow \mathbb{R} \) is an epimorphism!
Example 3.31: An interesting example of a universal construction arising in an “unnatural way” is that of the kernel in an Abelian category. An Abelian category, for our purposes, will be denoted $\mathcal{A}$ and can be thought of simply as a category of modules $\Lambda\text{Mod}$. The main properties and definitions can be found in [H-1970, § II.9].

Consider $I = \bullet \rightarrow \bullet$ and the subcategory $\mathcal{D}$ of $\mathcal{A}$ given by $O\mathcal{D} = O\mathcal{A}^I$ and

$$\mathcal{M}\mathcal{D} = \left\{ f \in \mathcal{M}\mathcal{A}^I \mid f = \begin{cases} x_2 & \rightarrow & 0 \\ 0 & \leftarrow & y_2 \\ x_1 & \rightarrow & y_1 \end{cases} \right\}.$$ 

We show that the couniversal construction that arises from the functor $P: \mathcal{A} \rightarrow \mathcal{D}$ given by $P(f: X \rightarrow Y) = \begin{cases} X & \rightarrow & 0 \\ 0 & \leftarrow & Y \\ X & \rightarrow & Y \end{cases}$ yields the kernel (in the categorical sense, see [H-1970]).

Consider the adjunction $\eta: P \dashv R$. We show that for $Y = (f: Y_1 \rightarrow Y_2 \in \mathcal{D})$, the morphism $\delta_{Y,1}: R Y \rightarrow Y_1$ is the kernel of $f$. Namely we show: for any $X \in \mathcal{A}$ and any map $\phi: X \rightarrow Y_1$ such that $f\phi = 0$ there exists a unique $\overline{\phi}: X \rightarrow R Y$ such that $\delta_{Y,1}\overline{\phi} = \phi$.

Consider a map $\phi: PX \rightarrow Y$. This is equivalent to considering a general $\phi_1: X \rightarrow Y_1$ such that $f\phi_1 = 0$. From the adjunction $\eta: \mathcal{A}(PX, Y) \xrightarrow{\sim} \mathcal{D}(X, R Y)$, we obtain a unique $\overline{\phi} = \eta(\phi): X \rightarrow R Y$ which satisfies $\delta\phi = P\overline{\phi}$. Interpreting this with the aid of the diagram

we see that $\delta_{Y,1}: R Y \rightarrow Y_1$ is the kernel of $f: Y_1 \rightarrow Y_2$. An entirely similar construction can be used to define a cokernel.

Example 3.32: Pull-backs and push-outs are a specific case of limits and colimits that we will use later on. Consider $I = \bullet \leftarrow \bullet \rightarrow \bullet$. The universal construction, i.e. colimit or left
adjoint, with respect to the constant functor \( P : \mathcal{C} \to \mathcal{C}^I \) defines the push-out in \( \mathcal{C} \). One frequently used example of a push-out is that of an amalgamated product in \( \text{Gp} \). Given

\[
\begin{array}{c}
N \xrightarrow{f_1} G \\
\downarrow f_2 \\
H
\end{array}
\]

the push out of this diagram is given by \( G \ast_N H = G \ast H / \langle f_1(n)(f_2(n))^{-1} \rangle_{G \ast H} \) with the canonical maps \( \varepsilon_1 : G \to G \ast_N H \) and \( \varepsilon_2 : H \to G \ast_N H \). We once again see how the colimit is a quotient object of the coproduct.

Now consider \( J = \bullet \to \bullet \leftarrow \bullet \). The couniversal construction with respect to the constant functor \( P : \mathcal{C} \to \mathcal{C}^I \) defines the pull-back.

**Example 3.33:** Our last example may be considered to be the motivating example for the definition of adjoint functors. It is in fact one of the very first things Kan mentions in his pioneering paper [K-1958, p. 294] on the subject. The example is the adjointness of \( - \otimes B \dashv \text{hom}_{\mathbb{Z}}(B, -) \).

Consider the category \( \text{Mod}_\Lambda \) of right \( \Lambda \)-modules where \( \Lambda \) is a ring with identity (not necessarily commutative). Then for any \( B \in \text{Mod}_\Lambda \), we may consider the functors \( - \otimes_{\Lambda} B : \text{Mod}_\Lambda \to \text{Ab} \) and \( \text{hom}(B, -) : \text{Ab} \to \text{Mod}_\Lambda \). We then see that

\[
\eta : \text{hom}_{\mathbb{Z}}(A \otimes_{\Lambda} B, C) \sim \text{hom}_{\Lambda}(A, \text{hom}_{\mathbb{Z}}(B, C))
\]

where \( \overline{\eta} := \eta(f : B \otimes A \to C) \) is defined by \( \overline{\eta}(a)(b) = f(b \otimes a) \) and extended by linearity. To see the equivalence, we define an inverse to \( \eta \) by \( \overline{\eta}^{-1} = \eta^{-1}(g : A \to \text{hom}_{\mathbb{Z}}(B, C)) \) where \( \overline{\eta}(b \otimes a) = g(a)(b) \) and \( \overline{\eta} \) is extended to all of \( B \otimes A \) by linearity.

4. **Universal (Co-)Universal Constructions**

The word “universal” is often times abused for many things within category theory. The fact that some universal constructions seem to defy generalization to a general category makes the use of the adjective “universal” in this situation questionable. An adjunction like \( - \otimes B \dashv \text{hom}_{\mathbb{Z}}(B, -) \) is an example that illustrates this. Another example is that of the kernel in example 3.1; it is only possible to talk about this construction in a category with a 0 object. We thus propose adjunctions should be called “universal constructions” only if the construction can be applied to an arbitrary category. Perfect examples of this universality are limits, colimits, initial objects and terminal objects.

We also suggest that if one uses the term “universal construction”, that one also states the class of categories to which it is universal to. For example, we propose one say that the kernel is a couniversal construction with respect to the class of Abelian categories instead of just saying that the kernel is a couniversal construction. We also suggest that the terminology “universal property” be made precise to clarify this matter. We define it in the course of our discussion below. Our viewpoint is that an adjunction determines a (co)-universal property in a category, but a universal construction should provide a universal property for any given category.

As such a change of language is unlikely to be widely accepted in the mathematical community, we content ourselves by introducing universal (co-)universal constructions which generalize constructions like (co-)limits and initial and terminal objects that meet the idealized notion of universality mentioned above.
**Definition 4.34:** Consider a (meta)functor $i : \text{Cat} \to \text{Cat}$ which preserves adjoints, i.e. if $F \dashv G$, then $iF \dashv iG$; consider a natural transformation $P^i : \text{id}_{\text{Cat}} \to i$. For any category $\mathcal{C}$, we define a left adjoint to $P^i_\mathcal{C} : \mathcal{C} \to i\mathcal{C}$ to be a universal universal construction, and a right adjoint to $P^i_\mathcal{C}$ to be a universal couniversal construction. We typically will abbreviate these as UUCs and UCCs.

One reason why the adjoint preserving property is included in this definition is because of the following useful proposition.

**Proposition 4.35:** Consider $F : \mathcal{C} \to \mathcal{D} \in \text{Cat}$. If $L \dashv F$ and a given UCC $(i, P^i)$ exists in both $\mathcal{C}$ and $\mathcal{D}$, then the UCC commutes in $\mathcal{D}$. That is, if $P^i_\mathcal{C} \dashv R^i_\mathcal{C}$ and $P^i_\mathcal{D} \dashv R^i_\mathcal{D}$, then $R^i_\mathcal{D} \circ iF \cong F \circ R^i_\mathcal{C}$. A similar result holds for all manner of permutations of left and right adjoints.

**Proof.** For the general result, the following diagram is helpful.

By proposition 22, we have $iL \circ P^i_\mathcal{D} \dashv R^i_\mathcal{D} \circ iF$ and $P^i_\mathcal{C} \circ L \dashv F \circ R^i_\mathcal{C}$. We see that $P^i_\mathcal{C} \circ L = iL \circ P^i_\mathcal{D}$ by the naturality of $P^i : \text{id}_{\text{Cat}} \to i$, and therefore, by proposition 21, there is a natural equivalence $R^i_\mathcal{D} \circ iF \cong F \circ R^i_\mathcal{C}$. The proof of the remaining parts follow analogously.

<table>
<thead>
<tr>
<th>$L \dashv F$</th>
<th>UUCs commute in $\mathcal{C}$, i.e. $L^i \circ iL \cong L \circ L^i_\mathcal{D}$</th>
<th>$L \dashv F$</th>
<th>UUCs commute in $\mathcal{D}$, i.e. $R^i_\mathcal{D} \circ iF \cong F \circ R^i_\mathcal{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F \dashv R$</td>
<td>UUCs commute in $\mathcal{D}$, i.e. $L^i_\mathcal{D} \circ iF \cong F \circ L^i_\mathcal{C}$</td>
<td>$F \dashv R$</td>
<td>UUCs commute in $\mathcal{C}$, i.e. $R^i_\mathcal{C} \circ iR \cong R \circ R^i_\mathcal{D}$</td>
</tr>
</tbody>
</table>

One can relax the existence of $R^i_\mathcal{D}$ to get that for $\mathfrak{X} \in i\mathcal{C}$, the object $F \circ R^i_\mathcal{C}(\mathfrak{X})$ satisfies the relevant couniversal property in $\mathcal{D}$. This is of importance in the later sections.

**Proposition 4.36:** Consider $F : \mathcal{C} \to \mathcal{D}$ and suppose $L \dashv F$. If the UCC $(i, P^i)$ exists in $\mathcal{C}$, i.e. there is $P^i_\mathcal{C} \dashv R^i_\mathcal{C}$, then for $\mathfrak{X} \in i\mathcal{C}$ the object $F \circ R^i_\mathcal{C}(\mathfrak{X})$ with $iF(\delta_\mathfrak{X})$ satisfies the couniversal property determined by $P^i_\mathcal{D}$ for $iF(\mathfrak{X})$. That is, for any $Y \in \mathcal{D}$ and any map $\phi : P^i_\mathcal{D}Y \to iF\mathfrak{X}$, there is a unique $\overline{\phi}$ making the following diagram commute.

$$
\begin{array}{c}
\text{PFR\mathfrak{X}} \\
\phi \\
\text{PY}
\end{array}
\xrightarrow{\text{PFR\mathfrak{X}}} iF\mathfrak{X}
$$
Proof. The proof is straightforward when one considers the relationship between the above diagram and

\[
\begin{array}{c}
\text{PRL} X \xrightarrow{\delta_X} X \\
\downarrow \phi' \\
\text{PLY} \\
\end{array}
\]

where \( \eta_{PY, X} : i\mathcal{C}iLPY, X \xrightarrow{\sim} i\mathcal{D}(PY, iFX) \).

We now give a few examples of UUCs and UCCs.

**Example 4.37:** The UUCs and UCCs that motivated the general definition are limits and colimits. They both arise from a small category \( I \) and considering the (meta)functor \( - : \text{Cat} \to \text{Cat} \) and the diagonal natural transformation \( P_- : \text{id}_{\text{Cat}} \to - \). Specifically, we have

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{F} & \mathcal{D} \\
- & \downarrow \mathcal{E} & \downarrow \mathcal{D} \\
\text{Cat} & \xrightarrow{F^l} & \mathcal{D}^l \\
\end{array}
\]

where \( F^l \) works "component-wise" on maps of a morphism \( \phi : X \to Y \in \mathcal{E}^l \). Such a \( \phi \) is composed of morphisms \( \phi_i : X_i \to Y_i \), for each \( i \in I \). Then \( F^l(\phi) : F^l(X) \to F^l(Y) \) is given by applying \( F \) to each \( \phi_i \). It is easy to verify that \( -^l \) preserves adjointness.

It is likewise simple to verify that \( P_- : \text{id}_{\text{Cat}} \to -^l \) given by the diagonal functor on each \( \mathcal{E} \) is a natural equivalence. That is, for any \( F : \mathcal{E} \to \mathcal{D} \), the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & \mathcal{D} \\
\downarrow P_{\mathcal{E}} & & \downarrow P_{\mathcal{D}} \\
\mathcal{E}^l & \xrightarrow{F^l} & \mathcal{D}^l .
\end{array}
\]

**Example 4.38:** The constant (meta)functors \( C_A \) for some \( A \in \text{Cat} \) provide the simplest possible \( i \) for constructing new UUCs and UCCs. Coming up with a \( P_- : \text{id}_{\text{Cat}} \to C_A \) seems to pose a greater challenge. One example that uses this constant (meta)functor is the construction of initial and terminal objects. These are constructed by taking \( i = C_2 \) where \( 1 = \bullet \) is the category with one object and one morphism, and \( P_{\mathcal{E}} : \mathcal{E} \to 1 \) defined by \( P_{\mathcal{E}}(X \to Y) = \text{id}_\bullet \). It is easy to verify that this satisfies all of the conditions of a UUC and UCC.

Other examples of UUCs and UCCs have not been easy to find. Perhaps relaxing the naturality of \( P_- \) is in order to fit the idea of universality better, as (co-)free objects do not seem to be UUCs or UCCs. The statement of Freyd’s Adjoint Functor Theorem provides a basis to conjecture that (co-)limits, initial and terminal objects are all of the possible UUCs and UCCs.

5. **"I HAVE A FUNCTOR, BUT DOES IT HAVE AN ADJUNCT?"**

The previous section offers some techniques to show adjoints don’t exist. Suppose we are considering \( F : \mathcal{E} \to \mathcal{D} \) and wish to know if it has a left or right adjoint. One method would be to see if a UUC (resp. UCC) exists for \( \mathcal{E} \) and determine whether \( F \) preserves the UUC (resp. UCC) or not. Another property to check is the (co-)solution set condition.
Definition 5.39: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and consider $Y \in \mathcal{D}$. A solution set for $Y$ is a set $\{X_i \in \mathcal{C} \mid i \in I\}$ and $\{f_i : Y \to FX_i \mid i \in I\}$ where $I$ is a set (yes, a set!) if: for any $X \in \mathcal{C}$ and any $\phi : Y \to FX$ there exists an $i$ and $\bar{\phi} : X_i \to X$ such that the following diagram commutes

\[
\begin{array}{ccc}
FX_i & \xrightarrow{f_i} & Y \\
\downarrow{F\bar{\phi}} & & \\
FX & \xrightarrow{\phi} & \\
\end{array}
\]

Remark 5.40: It is clear that if $L \dashv F$ then $\{FLY\}$ with $\{\varepsilon_Y : Y \to FLY\}$ is a solution set for $Y$. Thus for a left adjoint to $F$ to exist, the functor $F$ must satisfy the solution set condition.

Codefinition 5.41: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and consider $Y \in \mathcal{D}$. A cosolution set¹ for $Y$ is a set $\{X_i \in \mathcal{C} \mid i \in I\}$ and $\{f_i : FX_i \to Y\}$ where $I$ is a set if: for any $X \in \mathcal{C}$ and any $\phi : FX \to Y$ there exists an $i$ and $\bar{\phi} : X \to X_i$ such that the following diagram commutes

\[
\begin{array}{ccc}
FX_i & \xrightarrow{f_i} & Y \\
\downarrow{\phi} & & \\
FX & \xrightarrow{F\bar{\phi}} & \\
\end{array}
\]

Coremark 5.42: It is clear that if $F \dashv R$ then $\{FRY\}$ with $\{\delta_Y : FRY \to Y\}$ is a cosolution set for $Y$. Thus for a right adjoint to $F$ to exist, the functor $F$ must satisfy the cosolution set condition.

The following theorem due to Freyd provides a partial converse to the above remarks. The conditions are slightly idealized, however, and it will not be of much use to us. We state the theorem without proof; see [ML-1971, § V.6] or [M-1967, § V.3] for a proof.

Theorem 5.43: (Freyd’s Adjoint Functor Theorem) Suppose $\mathcal{C}$ is a small-complete category (that is, all limits for $I$ a small category exist) which has $\mathcal{C}(X,Y)$ a set for all objects $X, Y$. Then a functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if $F$ preserves all small limits and there is a solution set for all $Y \in \mathcal{D}$.

Question 5.44: Can we get a statement like this without restrictions on the category and just on the functor?

Example 5.45: In an introductory course on real analysis, one learns that $\mathbb{Q}$ is not complete, and $\mathbb{R}$ is the completion of $\mathbb{Q}$. From this, we conclude that the constant functor $P : \mathbb{Q} \to \mathbb{Q}^I$ does not have a left or right adjoint when $I$ is infinite. The category $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ of extended real numbers is complete and cocomplete. We may now ask if the canonical inclusion $\iota : \mathbb{Q} \to \overline{\mathbb{R}}$ has a left or right adjoint.

Let $Y = +\infty$. Then for any set $A \subseteq \mathbb{Q}$ the morphism sets $\mathbb{R}(\iota a, +\infty) = \emptyset$ for all $a \in A$. Thus the solution set condition fails and there is no left adjoint to $\iota$. Similarly, no right adjoint to $\iota$ exists by considering $Y = -\infty$.

Another way to state this is to first observe that $+\infty$ is the terminal object of $\overline{\mathbb{R}}$ and $-\infty$ is the initial object of $\overline{\mathbb{R}}$. If a left (resp. right) adjoint of $\iota$ were to exist, it would have to send the initial (resp. terminal) object of $\overline{\mathbb{R}}$ to the initial (resp. terminal) object of $\mathbb{Q}$. As $\mathbb{Q}$ does not have initial or terminal objects, no such adjoints can exist.

¹This terminology is not necessarily standard.
6. Adjoints to Contravariant Functors

We can interpret all of the results established above for contravariant functors \( F : \mathcal{C} \to \mathcal{D} \), i.e. a functor \( F : \mathcal{C}^{\text{op}} \to \mathcal{D} \). For sake of clarity, we write out what an adjunction \( F \dashv G \) looks like for a contravariant functor \( F \).

**Remark 6.46:** Consider \( F : \mathcal{C} \to \mathcal{D} \) a contravariant functor. An adjunction \( F \dashv G \), where \( G : \mathcal{D} \to \mathcal{C}^{\text{op}} \), is given by a natural equivalence

\[
\eta_{X,Y} : \mathcal{D}(FX,Y) \sim \mathcal{C}^{\text{op}}(X,GY) = \mathcal{C}(GY,X).
\]

**Example 6.47:** A nice, commonly encountered example of such an adjunction appears in the study of vector spaces. Let \( \text{Vect}_k \) be the category of all vector spaces over a field \( k \). Define \( D = \text{hom}(\_,-) : \text{Vect}_k \to \text{Vect}_k ; \) this is a contravariant functor. This recast in terms of \( \text{op} \)-categories and functors is \( D : \text{Vect}_k^{\text{op}} \to \text{Vect}_k \). We claim \( D^{\text{op}} \dashv D \), and also that \( D \not\dashv D^{\text{op}} \).

It is simple enough to verify that the following transformation is natural which establishes \( D^{\text{op}} \dashv D \).

\[
\eta_{V,W} : \text{Vect}_k^{\text{op}}(D^{\text{op}}(V),W) \sim \text{Vect}_k(V,D(W))
\]

\[
F^{\text{op}} : \text{Hom}(V,k) \to W \mapsto ((\eta_{V,W}F)(v))(w) = (F(w))(v).
\]

This adjunction tells us the useful fact that \( D \) sends UCCs in \( \text{Vect}_k^{\text{op}} \), i.e. UUCs in \( \text{Vect}_k \), to UUCs in \( \text{Vect}_k \). In particular, \( \text{hom}(\prod V_i,k) \cong \prod \text{hom}(V_i,k) \).

We now show that there is no adjunction \( D \dashv D^{\text{op}} \). If \( D \dashv D^{\text{op}} \) were the case, \( D \) would have to send UUCs in \( \text{Vect}_k^{\text{op}} \), i.e. UUCs in \( \text{Vect}_k \), to UUCs in \( \text{Vect}_k \). We particularly investigate what \( D \) does on coproducts in \( \text{Vect}_k \).

Consider \( I = \mathbb{N} \) a countable indexing category, and the diagram \( \mathfrak{X} : I \to \text{Vect}_k \) given by \( \mathfrak{X}(i) = k \) for all \( i \in I \). Then \( \prod \mathfrak{X} = \prod \mathbb{N} k \) is in fact the free \( k \) vector space on an uncountable number of generators. This is easy to establish using a cardinality argument and the fact that Zorn’s lemma implies \( \prod \mathfrak{X} \) does have a basis. It therefore follows that \( D(\prod \mathfrak{X}) = \text{hom}(\prod \mathfrak{X},k) \) is the product of an uncountable number of copies of \( k \). This is clearly not isomorphic to \( \prod D^I \mathfrak{X} \), the sum of a countable number of copies of \( k \). Therefore there is no adjunction \( D \dashv D^{\text{op}} \).

This example can be used as justification for considering the existence of an adjoint between categories as saying the categories are equivalent. The dual functor is an isomorphism for finite dimensional vector spaces, but then fails for infinite dimensional vector spaces. However, there is still an adjoint relationship \( D^{\text{op}} \dashv D \).

If we wish to consider those functors which have adjoints as a new kind of equivalence of categories, it is then of great importance to understand UUCs and UCCs as they are preserved under adjoints, and thus possess some of the intrinsic structure of a category.

7. Functors in Topology

The question that motivates this section is, “Does the de-Rham cohomology functor, in any of its forms, have an adjoint?” One might anticipate that it does, as it sends coproducts to products—a hallmark characteristic of what it means for there to be an adjoint! To proceed, we verify that Freyd’s adjunction theorem does not apply.

First off, \( \text{SimMan} \) is not cocomplete. We will show that the pushout of
The reader is cautioned from using intuition about colimits in \[\text{Top}\] when thinking about colimits in \[\text{SmMan}\] and \[\text{Man}\]. We have the following example that shows we need to be careful!

**Example 7.48:** Consider the diagram

\[
\begin{array}{ccc}
\{0\} & \to & \mathbb{R} \\
\downarrow \ & \ & \downarrow \\
\mathbb{R} & \to & \mathbb{R}
\end{array}
\]

It is clear that the colimit (push-out) of this diagram in \[\text{Top}\] is the line with two origins. However, in \[\text{Man}\] and \[\text{SmMan}\] the colimit is just \(\mathbb{R}\) itself. This follows as any map \(f : \mathbb{R} \to X\) is uniquely determined by the values of \(f\) on \(\mathbb{R} \setminus \{0\}\) when \(X\) is Hausdorff. Consider \(X \in \text{Man}\) and maps \(\phi_1, \phi_2 : \mathbb{R} \to X\) which make the diagram commute below, i.e. \(\phi_1|_{\mathbb{R}\setminus\{0\}} = \phi_2|_{\mathbb{R}\setminus\{0\}}\). Then by the Coincidence Theorem, \(\phi_1 = \phi_2\) and hence taking \(\phi = \phi_1 = \phi_2\) shows that \(\mathbb{R}\) is the colimit in \[\text{Man}\], which is not homeomorphic to the line with two origins.

In order to establish that \[\text{SmMan}\] is not cocomplete, we will utilize the contravariant functor \(C^\infty(\_, \mathbb{R}) : \text{SmMan} \to \text{Alg}_{\mathbb{R}}\). The main property it posses is that it sends those colimits which exist to limits. As limits are better understood in \[\text{Alg}_{\mathbb{R}}\], we can compute the limit of \(C^\infty(\mathbb{R} \leftarrow \{0\} \to \mathbb{R})\) and see if that \(\mathbb{R}\)-algebra is in the image of \(C^\infty(\_, \mathbb{R})\). In order to do this efficiently, we utilize the following result about ideals in rings of the form \(C^\infty(M, \mathbb{R})\) where \(M \in \text{SmMan}\) and defer the proof to the end of the paper.

**Proposition 7.49:** Let \(M \in \text{SmMan}\). For a point \(p \in M\), define the \(I_p\) to be the ideal \(I_p = \{f \mid f(p) = 0\}\) in \(C^\infty(M, \mathbb{R})\). This ideal is maximal, and the product ideals \(I_p^n = \langle g_1 \cdots g_n \mid g_i \in I_p\rangle\) have the following description

\[
I_p^n = \left\{ f : \frac{\partial^{\vert \alpha \vert}}{\partial x^\alpha} f(p) = 0 \forall \alpha, \vert \alpha \vert \leq k - 1 \right\}
\]

\[
= \{ f : f \text{ has } k - 1 \text{ order contact with } 0 \}
\]

\[
= \{ f : \text{the Taylor series of } f \text{ has no terms of degree } \leq k - 1 \}
\]

where \(\alpha \in \mathbb{N}_0^n\) and \(\vert \alpha \vert = \sum_{i=1}^n \alpha_i\) is the sum of the entries. Also \(x^\alpha = \prod x_i^{\alpha_i}\) for \(x \in \mathbb{R}^n\).

The above shows that the ideals \(I_p^n\) contain local information. Define two functions \(f, g \in C^\infty(M, \mathbb{R})\) to be locally equivalent at \(p\) if there exists an open set \(U \subseteq M\) with \(p \in U\) such that \(f|_U = g|_U\); we write \(f \sim_p g\) and denote equivalence classes as \([f]_p\)
which are called germs. The collection of all germs forms an $\mathbb{R}$-algebra under the canonical operations which we denote by $C^\infty_p(M, \mathbb{R})$. Define $m_p = \{ f | f(p) = 0 \}$. Consider the canonical projection $\phi: C^\infty(M, \mathbb{R}) \rightarrow C^\infty_p(M, \mathbb{R})$, then $I^n_p = \phi^{-1}(m^n_p)$.

**Remark 7.50:** If one restricts ones attention to the category $\mathcal{A}\text{Man}$, the above proposition can be gotten much cheaper! It is perhaps instructive for the reader to work through this case to see how the machinery works.

**Proposition 7.51:** The diagram $\mathbb{R} \leftarrow \{0\} \rightarrow \mathbb{R}$ does not have a colimit in $\text{SmMan}$.

**Proof.** Suppose that the colimit does exist and call it $M$, and denote the maps of the unit as $\varepsilon_1$ and $\varepsilon_2$ as seen in the following diagram

\[
\begin{array}{ccc}
\mathbb{R} \setminus \{0\} & \xrightarrow{\varepsilon_1} & \mathbb{R} \\
\wedge & & \downarrow \varepsilon_2 \\
\mathbb{R} & \xleftarrow{\varepsilon_1} & M.
\end{array}
\]

As $C^\infty(-, \mathbb{R})$ sends colimits to limits, we have

\[
C^\infty(M, \mathbb{R}) \cong \{ (f_1, f_2) | f_1 \in C^\infty(\mathbb{R}, \mathbb{R}), f_1(0) = f_2(0) \}.
\]

Denote $x := \varepsilon_1(0) = \varepsilon_2(0) \in M$. We then investigate the local properties of $M$ at $x$ by studying the ideals $I^n_x/I^n_x+1$. We demonstrate that $\dim \mathbb{R} I^n_x/I^n_x+1 = 2$ for all $n$ which is impossible for a smooth manifold.

We first establish the dimension computation of $I^n_p/I^n_p+1$ in $C^\infty(\mathbb{R}, \mathbb{R})$ with $p \in \mathbb{R}$. If $f \in I^n_p$, then $f - f^{(n)}(p)(x - p)^n \in I^n_p+1$. Hence $I^n_p/I^n_p+1 = f((x - p)^n)_{x=p}$.

We now compute $\dim \mathbb{R} I^n_x/I^n_x+1$ in $C^\infty(M, \mathbb{R})$. We make the convention that $I^n_p := I^n_p$ in $C^\infty(\mathbb{R}, \mathbb{R})$. Observe that $I^n_x = (f_{11} \cdots f_{1n}, f_{21} \cdots f_{2n}) | f_{ij} \in J_0)$. Therefore, the quotient $I^n_x/I^n_x+1 = ((x^n, 0), (0, x^n))_{x=p}$. We conclude $\dim \mathbb{R} I^n_x/I^n_x+1 = 2$.

For a general manifold $X$, the computation of $\dim \mathbb{R} I^n_p/I^n_p+1$ is local, so it suffices to carry out the computation at a point $p \in X$ through the local coordinate charts in $\mathbb{R}^k$, which we carry out now. Consider $f \in I_p \subseteq C^\infty(\mathbb{R}^k, \mathbb{R})$. Then $f - \sum \frac{\partial f}{\partial x_i}(p) \cdot (x_i - p_i) \in I^n_p$, and it is clear that $(x_i - p_i) - (x_j - p_j) \notin I^n_p$ from which we conclude $I_p/I^n_p = (x_1, \ldots, x_k)_{x=p}$.

We now verify that if $X \in \text{SmMan}$ and $\dim \mathbb{R} I^n_p/I^n_p+1 = 2$ (i.e. $X$ is locally 2-dimensional at $p$) that $\dim \mathbb{R} I^2_p/I^3_p = 3$ from which we conclude that $M \notin \text{SmMan}$. Since the computation of $\dim \mathbb{R} I^n_p/I^n_p$ doesn’t depend on if it is done in the manifold $X$ or in $\mathbb{R}^2$, we carry out the computation for $\mathbb{R}^2$. We see that for $f \in I^2_p$,

\[
f - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(p) \cdot (x - p_1)^2 - \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(p) \cdot (y - p_2)^2 - \frac{\partial^2 f}{\partial xy}(p) \cdot (x - p_1)(x - p_2) \in I^3_p.
\]

It is therefore evident that $I^2_p/I^3_p = (x^2, y^2, xy)_{x=p}$; hence the result. ▲

Therefore we cannot use Freyd’s adjoint functor theorem to prove the existence or nonexistence of an adjoint to $H^*_d\text{R}$. There are a few ways to see that it does not have a left or right adjoint, however: we can show the (co-)solution set conditions do not hold, or compute $H^*_d\text{R}$ of limits and colimits and see if they are not sent to the corresponding colimits and limits. We carry out both methods to see that no $H^*_d\text{R} : \text{SmMan} \rightarrow \text{Vect}_{\mathbb{R}}$ has an adjoint, and also that $H^*_d\text{R} : \text{SmMan} \rightarrow \text{Alg}_{\mathbb{R}}$ does not have an adjoint.

Recall that the Künneth formula tells us that $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$. Thus the case when $X = Y = S^1$ tells us the cohomology of the torus, which is clearly not
$H^*(S^1) \times H^*(S^1)$. Therefore there is no right adjoint to $H^*$ as a functor to $\text{Alg}_R$. One can construct similar examples to see that no $H^i$ has a right adjoint either.

To show that $H^*$ does not have a left adjoint, we need to investigate colimits that are more complicated than just coproducts. The Mayer-Vietoris sequence on two open sets tells us how to compute the de Rham cohomology of nice push-out diagrams. The long exact sequence can in fact tell us how far away $H^i$ of a push-out is from being the pull-back. The Mayer-Vietoris sequence tells us that if $\{U, V\}$ is an open cover of a smooth manifold $M$, then

$$U \cap V \stackrel{i_U}{\longleftarrow} U \coprod V \stackrel{\pi}{\longrightarrow} M$$

induces a sequence

$$\Omega^*(M) \longrightarrow \Omega^*(U) \prod \Omega^*(V) \longrightarrow \Omega^*(U \cap V)$$

and then a long exact sequence of cohomology

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U) \times H^0(V) \longrightarrow H^0(U \cap V) \longrightarrow H^1(M) \longrightarrow H^1(U) \times H^1(V) \longrightarrow H^1(U \cap V) \longrightarrow \cdots$$

Let us write $M = \text{colim} (U \leftarrow U \cap V \rightarrow V)$. If $H^i$ were to send this colimit to a limit, the following sequence must be exact

$$0 \longrightarrow H^i(M) \longrightarrow H^i(U) \times H^i(V) \longrightarrow H^i(U \cap V)$$

that is, $H^i(M) \cong \{ (f, g) \in H^i(U) \times H^i(V) \mid f|_{U \cap V} = g|_{U \cap V} \}$ which is clearly the description of $\text{lim} (H^i(U) \to H^i(U \cap V) \leftarrow H^i(V))$.

We see by the Mayer-Vietoris sequence, that the exactness will fail in general for $i > 0$, and what controls it is the connecting map $\theta$. A concrete example of this arises by considering the following two set cover of the torus

![Diagram of a torus]

To show that $H^0$ has no left adjoint, we show that the solution set condition fails. In our definition for the solution set condition, take $Y = \mathbb{R}$ and consider a general set $\{X_i\}$ of smooth manifolds with maps $\{f_i : \mathbb{R} \to H^0(X_i)\}$. Each space $X_i$ decomposes as a disjoint union of its connected components, which we write $X_i = \coprod_{\mathcal{J}} X_{ij}$ where $\mathcal{J}$ is some set. Then $H^0(X_i) = \coprod_{\mathcal{J}} \mathbb{R}e_{ij}$. We now take $X$ to be a point, or any smooth manifold with only one connected component, so that $H^0(X) = \mathbb{R}$. If $f_i(e_1) = \sum_{j=1}^{k} r_{ji} e_{ij}$ for $r_{ji} \neq 0$ and $k \geq 1$, take $\phi(e_1) = (2 \cdot \prod r_{ji}) e_1$. Thus for any map $\bar{\phi} : X \to X$, the space $X$ must be
sent entirely into one connected component $X_{it}$. Thus $H^0\phi(e_{it}) = e_1$ and $H^0\phi(e_{ij}) = 0$ for all other $j$. Hence

$$H^0\phi(f_i(e_1)) = \begin{cases} r_1 e_1 & \text{or} \\ 0 & \neq 2 \cdot \prod r_{ji} e_1. \end{cases}$$

Now if $f_i(e_1) = 0$, simply take $\phi(e_1) \neq 0$. Thus it is clear that the solution set condition does not hold, hence the result.

**Remark 7.52:** Behind the proof that not all colimits exist in SmMan is a very general technique that relates to our discussion about the completion of $\mathbb{Q}$. If we want to show that a subset $A \subseteq \mathbb{Q}$ does not have (co-)product in $\mathbb{Q}$, it is much easier to construct the completion $\mathbb{R}$ of $\mathbb{Q}$ and see what the (co-)product of $\iota(A)$ is there. Since $\iota: \mathbb{Q} \hookrightarrow \mathbb{R}$ preserves all of those (co-)products which do exist in $\mathbb{Q}$, we can see what the $\prod \iota(A)$ is, and see if it is in $\mathbb{Q}$ or not. We really did just this to show that not all colimits exist in SmMan, just with more complicated machinery and objects. We knew that $\text{Alg}_{\mathbb{R}}$ had all limits, and knew that $C^\infty(\_ , \mathbb{R})$ sent those colimits which do exist to limits. We then investigated the ring $C^\infty(M, \mathbb{R})$ to see if it was in the image of $C^\infty(\_ , \mathbb{R})$ and saw that it wasn’t; hence the colimit did not exist in SmMan.

One can ask the same question for limits in SmMan. Instead of using $C^\infty(\_ , \mathbb{R})$, one can use the concept of a Hausdorff Frölicher space to carry out the method. The investigation of Frölicher spaces is left to another paper. See http://ncatlab.org for more details on the construction of Frölicher spaces.

### 8. Ideals in $C^\infty(M, \mathbb{R})$

We now prove proposition 49.

**Proposition 8.53:** Let $M \in \text{SmMan}$. For a point $p \in M$, define the $I_p$ to be the ideal $I_p = \{ f \mid f(p) = 0 \}$ in $C^\infty(M, \mathbb{R})$. This ideal is maximal, and the product ideals $I_p^n = \langle g_1 \cdot \ldots \cdot g_n \mid g_i \in I_p \rangle$ have the following description

$$I_p^n = \{ f : \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) = 0 \forall \alpha, |\alpha| \leq k - 1 \}$$

$$= \{ f : f \text{ has } k - 1 \text{ order contact with } 0 \}$$

$$= \{ f : \text{the Taylor series of } f \text{ has no terms of degree } \leq k - 1 \}$$

where $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is the sum of the entries. Also $x^\alpha = \prod x_i^{\alpha_i}$ for $x \in \mathbb{R}^n$.

In particular, we show:

1. The ideal $I_p^2$ has the above description in $C^\infty(\mathbb{R}^k, \mathbb{R})$;
2. By induction, the ideals $I_p^n$ admit the above description in $C^\infty(\mathbb{R}^k, \mathbb{R})$;
3. Using partitions of unity, we obtain the general result in $C^\infty(M, \mathbb{R})$.

**Proof.**
1. We first prove that if $f \in I_p$ and $\frac{\partial f}{\partial x_i}(p) = 0$ for all $1 \leq i \leq k$, then $f = \sum g_1, g_2 \in I_p^2$. Define $\phi(t) = f(p + t(x - p))$ for a general $x \in \mathbb{R}^k$. Then

$$f(x) = \phi(1) - \phi(0) = \int_0^1 \frac{d\phi}{dt}(t) dt$$

$$= \int_0^1 \sum_{i=1}^k (x_i - p_i) \cdot \frac{\partial f}{\partial x_i}(p + t(x - p)) dt$$

$$= \sum (x_i - p_i) \cdot \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt.$$ 

It is clear that defining $g_1(x) = (x_i - p_i)$ and $g_2(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt$ yields the desired decomposition as $g_2(p) = \int_0^1 \frac{\partial f}{\partial x_i}(p) dt = \int_0^1 0 dt = 0$.

Showing that $f \in I_p^2$ has all first order derivatives vanishing at $p$ is a straightforward application of the product rule.

2. The general case follows by induction. Suppose $f$ satisfies $\frac{\partial |\alpha|}{\partial x^\alpha}(p) = 0$ for all $|\alpha| < n$. We then show that

$$f(x) = \sum_{|\alpha|=n-1} (x-p)^\alpha \int_{[0,1]^{n-1}} \left( t_{n-1}^{n-2} t_{n-2}^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^n}(p + t_{n-2} \cdots t_{n-1}(x - p)) \right) dV$$

which is then in $I_p^n$.

Assume the result holds for $n$ and we show that it is true for $n + 1$. Under the assumption that $f$ satisfies $\frac{\partial |\alpha|}{\partial x^\alpha}(p) = 0$ for all $|\alpha| < n + 1$ we have

$$f(x) = \sum_{|\alpha|=n-1} (x-p)^\alpha \int_{[0,1]^{n-1}} \left( t_{n-1}^{n-2} t_{n-2}^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^n}(p + t_{n-2} \cdots t_{n-1}(x - p)) \right) dV$$

It is then clear that

$$\frac{\partial}{\partial x_i} \left( \int_{[0,1]^{n-1}} \left( t_{n-1}^{n-2} t_{n-2}^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^n}(p + t_{n-2} \cdots t_{n-1}(x - p)) \right) dV \right) (p)$$

$$= \int_{[0,1]^{n-1}} 0 dV = 0.$$

Hence we may apply the result obtained for $n = 2$ to each integral expression to get the desired representation of $f$.

3. The general results follows fairly easily from the above result using partitions of unity to spread the local result to the whole manifold. Specifically, let $U \subseteq M$ be an open set about $p$ which is homeomorphic to $\mathbb{R}^k$ through a local chart $\psi : U \to \mathbb{R}^k$. It is easy to see that if $f : M \to \mathbb{R} \in I_p^n$, then $f$ satisfies the partial derivative condition.

We now suppose $f : M \to \mathbb{R}$ satisfies $\frac{\partial |\alpha|}{\partial x^\alpha}(p) = 0$ for all $|\alpha| < n$ such that $|\alpha| < n$. In this case, $f \circ \psi^{-1}$ satisfies the derivative conditions at $\psi(p)$ so we may use the result proved above to obtain $f \psi^{-1} = \sum h_{i1} \cdots h_{in}$ where each $h_{ij}(\psi(p)) = 0$, and thus,

$$f|_U = \sum h_{i1} \circ (h_{i1} \circ \psi) \cdots (h_{in} \circ \psi).$$

We denote $h_{ij} \circ \psi = g_{ij}$.
To extend each $g_{ij}$ to all of $M$, we use a partition of unity $\{\phi_U, \phi_V\}$ subordinate to $\{U, V\}$ where $V = M \setminus W$ where $W \subset U$ is closed with $p \in W$. Then we compute

$$f = (\phi_U + \phi_V)^n \cdot f = \sum_{\ell=0}^{n} \binom{n}{\ell} \phi_U^{n-\ell} \phi_V^{\ell} f.$$ 

We can make sense out of this by observing $\phi_V$ has support in $V$, and thus by construction $\phi_V \in I_p$. Thus $\phi_V^n \cdot f \in I_p^n$. All other terms $\phi_V^{n-\ell} \phi_U^\ell f$ with $\ell \neq 0$ have support in $U \cap V$ where the description $f = \sum g_{ij}$ holds. Thus

$$\phi_V^{n-\ell} \phi_U^\ell f = \phi_V^{n-\ell} \phi_U^\ell \left( \sum g_{i1} \cdots g_{in} \right) = \sum \phi_U^{n-\ell} \phi_V^\ell g_{i1} \cdots g_{in} = \sum (\phi_U g_{i1})(\phi_U g_{i2}) \cdots (\phi_U g_{i(n-\ell-1)})(\phi_U g_{i(n-\ell)} \cdots g_{in}) \phi_U^\ell$$

and it is evident since $\phi_U$ has support in $U$ that each $(\phi_U g_{ij})$ is a well defined smooth function on $M$ with $(\phi_U g_{ij})(p) = 0$; hence the result.

\[\square\]

References


[MO1-2010] Ideals in the ring of smooth endomorphisms of the real line

[MO2-2010] Colimits in the Category of Smooth Manifolds