Periodic Orbits on Non-obtuse Edge-tessellating Polygons

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Imagine standing at a frictionless billiard table. Given a natural number $n$, how many ways can you throw a billiard ball so that it bounces around the table exactly $n$ times, returns to its initial position, and repeats?
Our Initial Research Goals:

• For edge tessellating polygons, classify and count their periodic orbits.
• Uncover the underlying connection between integer partitions into parts, and periodic orbits.
Previous Results (equilateral)

- Classes of orbits are identified by the coordinates of their *unfolding*.
- Equilateral triangles tiles the plane making rhombic coordinates.
  - Period is dependent on its \((x,y)\) coordinates: 
    \[2n = 2x + 2y\]
- Counting: bijection between all 2n-periodic orbits and partitions of \(n\) into 2’s and 3’s.
Les Partitions

• Any integer \( n \) can be partitioned into integers \( a \) and \( b \):
  
  – For \( n=12 \), \( a=3 \), \( b=2 \), we have 3 distinct partitions:
    
    \[
    12 = 2+2+2+2+2+2 = 2+2+2+3+3 = 3+3+3+3
    \]

• In general, the number of partitions will be the coefficient of the \( x^n \) term of the generating function:

\[
 f(x) = \frac{1}{(1 - x^a)(1 - x^b)}
\]
Edge-Tessellating Shapes Studied

- 30-60-90 Triangle
- 30-30-120 Triangle
- Rectangles
- Isosceles Right Triangle
Rectangles

• Tiling generates a regular grid.
• A class of orbits can be identified by its unfolding’s endpoint. Periodic orbits correspond precisely to points with even coordinates
  – The \((2x,2y)\) coordinates of an endpoint indicate its period:
    \[
    2n = 2x + 2y
    \]
• Counting: There are \(n\) positive integer solutions to the equation \(2x+2y=2n\).
Isosceles Right Triangle

- Square-grid structure with all orbits in the region $[45^\circ, 90^\circ]$
  - Endpoint of unfolding defines period:
    \[ 2n = 2x + 2y \]
- Counting: bijection between all $2n$-periodic orbits and partitions of $n$ into 2’s and 3’s
30-60-90 Triangle

- Coordinate system is formed from rectangles in region $[60^\circ, 90^\circ]$.
  - Endpoints of unfoldings define period:
    \[ 2n = x + 3y \]

- Counting: Bijection between all $2n$-periodic orbits and partitions of $n$ into 3’s and 5’s.
Underlying Rationale for Bijections Between Orbits and Partitions
Equilateral Triangle

- The two smallest orbits of the equilateral triangle are period 4 and period 6 ($v_4, v_6$)
- To count the orbits of this triangle, there is a bijection between $2n$-periodic orbits and partitions of $2n$ into 4’s and 6’s (or $n$ into 2’s and 3’s)
Isosceles Right Triangle

• The two smallest orbits are period 6 and period 4 ($v_6$, $v_4$).

• To count the orbits of this triangle, there is a bijection between $2n$-periodic orbits and partitions of $2n$ into 6’s and 4’s (or $n$ into 3’s and 2’s)
A New Method For Classifying and Counting Periodic Orbits (Conjecture)

• The two smallest orbits of a shape are the basis vectors which form an orbit space for the shape over \( \mathbb{Z} \). Any orbit has a one-to-one correspondence with some pair of integer coefficients \((a, b)\) times the basis orbits.

\[
v_z = av_x + bv_y
\]
Counting Period

• The *period* of any orbit is given by in terms of the integer pair.

\[ 2n = z = ax + by \]

• There is a bijection between 2n-periodic vectors and partitions of 2n into x and y.
30-60-90 Triangle

- The two smallest orbits are period 6 and period 10 ($v_6$, $v_{10}$).
- To count the orbits of this triangle, there is a bijection between $2n$-periodic orbits and partitions of $2n$ into 6’s and 10’s (or $n$ into 3’s and 5’s)
Relationship between 60-60-60 and 45-45-90

A Linear Transformation between their orbit spaces.
Transformation Example

Period 6
30-30-120 Triangle

• Roadblock: A unique singular where three 120 angles intersect vs. where six 30 angles intersect.

• Period depends on both angle and position.

• What defines an equivalence class?
Problems for the Future

• Develop the Orbit Space conjecture on the shapes we’ve studied.

• How do we handle the shapes with obtuse angles? (the 30-30-120 problem)

• Can these methods generalize to other triangles? Even more general shapes?