Prescribed Sets of Roots of Polynomial Equations Over $M_k(\mathbb{C})$

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Some Standard Theorems

Some Facts about Polynomial Equations over $\mathbb{C}$
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Vieta Theorem:
For any $x_1, x_2, \ldots, x_n \in \mathbb{C}$ (not necessarily distinct) there is a unique polynomial over $\mathbb{C}$

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

$$= (x - x_1)(x - x_2)\ldots(x - x_n)$$

such that the equation $f(x) = 0$ has roots $x_1, \ldots, x_n$ (counting multiplicities)
Theorems continued..

coefficients are given by

\[ a_{n-1} = (-1)^i \sum_{j_1 \ldots j_i} x_{j_1} \ldots x_{j_i} \]
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Fundamental Theorem of Algebra:

\[ x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \]

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Fundamental Theorem of Algebra:

\[ x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \]

has a root over \( \mathbb{C} \)
in fact, it has exactly \( n \) roots.
Question: What can we say about

\[ X^n + A_{n-1}X^{n-1} + \ldots + A_1X + A_0 = 0 \]

over \( M_k(\mathbb{C}) \)?
Polynomial Equations over $M_k(\mathbb{C})$

Differences to take into account:
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- multiplication in $M_k(\mathbb{C})$ is not commutative
Polynomial Equations over $M_k(\mathbb{C})$

Differences to take into account:

- multiplication in $M_k(\mathbb{C})$ is not commutative
- not all matrices in $M_k(\mathbb{C})$ are invertible
Analogues of the Vieta Theorem

For the case of $k=2$, $n=2$ consider $X_1, X_2 \in M_k(\mathbb{C})$ roots of the equation

$$X^2 + A_1X + A_0$$

It can be shown that if $(X_1 - X_2)$ is invertible, denoting $y_1 = X_1$, $y_2 = (X_1 - X_2)X_2(X_1 - X_2)^{-1}$ we can solve for the coefficients and obtain $A_1 = -(y_1 + y_2)$, $A_0 = y_2 y_1$
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This generalization can be extended to equations of degree \( n \) (due to Gelfand and Retakh)
However, it would be nice to understand a more general situation without the requirement of invertibility.
Compatible sets of Matrices over \( \mathbb{C} \)

**Definition:**
We call a set *Compatible* if there exists an \( n^{th} \) degree polynomial equation:
\[
X^n + A_{n-1}X^{n-1} + \ldots + A_0
\]
with \( X_1, X_2, \ldots, X_n \) as its roots.
some examples for \( n=2, \ k=2 \)

Some Examples:
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There is no equation

$$X^2 + A_1 X + A_0 = 0$$

such that

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

are roots, so \( \{X_1, X_2\} \) is **not** Compatible
Examples Continued...

While for

\[ X_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]

\( \{X_1, X_2\} \) is *Compatible* since \( X_1, X_2 \) are solutions (in fact unique solutions) to

\[ X^2 + X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \]
Our Project

Proposed Questions:

- Is there a generalized way to determine whether a set \( \{X_1, \ldots X_n\} \) of matrices over \( \mathbb{C} \) is Compatible?

- How many such sets can be characterized?

- Given a set \( \{X_1, \ldots X_i\} \in M_k(\mathbb{C}) \) which is not compatible, can one always find \( X_{i+1} \in M_k(\mathbb{C}) \) such that the set \( \{X_1, \ldots X_i, X_{i+1}\} \) is compatible?
Methods

Since invertibility and commutativity become an issue when looking for solutions, determining compatibility came down to looking at particular row space requirements.
Degree 2 case

Given the equation:

\[ X^2 + A_1 X + A_0 = 0, \]

\[ X, A_i \in M_k(\mathbb{C}) \]

we want to determine all solution sets \( \{X_1, X_2\} \)
Degree Two Case

Result: \(X_1, X_2\) are roots of \(X^2 + A_1 X + A_0 = 0\) for some \(A_1, A_0\) iff the row space of \((X_1 - X_2)X_2\) is contained in the row space of \((X_1 - X_2)\)
Question: Given \( \{X_1, X_2\} \) not compatible, is it possible to find \( X_3 \) such that \( \{X_1, X_2, X_3\} \) is compatible?
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\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

These two matrices have no degree two equation for which they are compatible. However,
\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

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\begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & 0
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0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

\[X^3 + \begin{pmatrix}
0 & 0 \\
-1 & 1
\end{pmatrix}X^2 + \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}X = 0\]
Finding Sets of Compatible matrices, cubic case

Looking for \( \{X_1, X_2, X_3\} \) satisfying
\[
X^3 + A_2X^2 + A_1X + A_0
\]
requires solving a system of equations with 3 unknowns.
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Looking for \( \{X_1, X_2, X_3\} \) satisfying
\[
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\]
requires solving a system of equations with 3 unknowns. This is easier to work with given certain "nice conditions".
Notation

Let $X_1 = X_1 - X_2$, $X_2 = X_1^2 - X_2^2$, $X_3 = X_1^3 - X_2^3$

$Y_1 = X_2 - X_3$, $Y_2 = X_2^2 - X_3^2$, $Y_3 = X_2^3 - X_3^3$
Notation

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Assuming $X_1$ invertible
Let \( X_1 = X_1 - X_2, \) \( X_2 = X_1^2 - X_2^2, \) \( X_3 = X_1^3 - X_2^3 \)

\( Y_1 = X_2 - X_3, \) \( Y_2 = X_2^2 - X_3^2, \) \( Y_3 = X_2^3 - X_3^3 \)

Assuming \( X_1 \) invertible

substitution and manipulation yields:

\[
A_2(X_2X_1^{-1}Y_1 - Y_2) = -X_3X_1^{-1}Y_1 + Y_3
\]
Let $X_1 = X_1 - X_2$, $X_2 = X_1^2 - X_2^2$, $X_3 = X_1^3 - X_2^3$

$Y_1 = X_2 - X_3$, $Y_2 = X_2^2 - X_3^2$, $Y_3 = X_2^3 - X_3^3$

Assuming $X_1$ invertible

substitution and manipulation yields:

$$A_2(X_2X_1^{-1}Y_1 - Y_2) = -X_3X_1^{-1}Y_1 + Y_3$$

So, in order for $X_1, X_2, X_3$ to be roots, the row space of

$$-X_3X_1^{-1}Y_1 + Y_3$$

must be contained in the rowspace of

$$X_2X_1^{-1}Y_1 - Y_2$$.

However, this is still rather ambiguous.
More Specific Examples

Work with more specific examples with "nice conditions". We return to our original $X_i$ notation. Let $X_1, X_2, X_3$ be roots so that $X_2 - X_3 = I$ thus $X_2 = X_3 + I$
More Specific Examples

Work with more specific examples with "nice conditions". We return to our original $X_i$ notation. Let $X_1, X_2, X_3$ be roots so that $X_2 - X_3 = I$ thus $X_2 = X_3 + I$

want to solve:

1. $A_2(X_1^2 - X_3^2) + A_1(X_1 - X_3) = -X_1^3 + X_3^3$
2. $A_2(X_2^2 - X_3^2) + A_1(X_2 - X_3) = -X_2^3 + X_3^3$
3. $A_2(X_3^2) + A_1(X_3) + A_0 = -X_3^3$
Example continued

Using Gaussian elimination..

\[
\begin{pmatrix}
A_2 & A_1 & A_0 \\
\end{pmatrix}
\begin{pmatrix}
X_1^2 - X_3^2 & X_2^2 - X_3^2 & X_3^2 \\
X_1 - X_3 & X_2 - X_3 & X_3 \\
0 & 0 & I \\
\end{pmatrix}
= \\
\begin{pmatrix}
-X_1^3 + X_3^3 & -X_2^3 + X_3^3 & -X_3^3 \\
\end{pmatrix}
\]
and substituting $X_2 = X_3 + I$ we can reduce this to

\[
\begin{pmatrix} A_2 & A_1 & A_0 \end{pmatrix} \begin{pmatrix} X_2^2 - X_3^2 - (I + 2X_3)(X_1 - X_3) & I + 2X_3 & X_3^2 \\ 0 & I & X_3 \\ 0 & 0 & I \end{pmatrix} = 
\begin{pmatrix} -X_1^3 + X_3^3 - (X_3^2 + (I + 3X_3)(I + 2X_3)(X_1 - X_3) & -X_2^3 + X_3^3 & -X_3^3 \end{pmatrix}
\]
Thus we are left to solve:

\[ A_2(X_1^2 - X_3^2 - (I + 2X_3)(X_1 - X_3)) = -X_1^3 + X_3^3 - (X_3^2 + (I + 3X_3)(I + 2X_3)(X_1 - X_3) \]
Thus we are left to solve:
\[ A_2(X_1^2 - X_3^2 - (I + 2X_3)(X_1 - X_3)) = -X_1^3 + X_3^3 - (X_3^2 + (I + 3X_3)(I + 2X_3)(X_1 - X_3) \]

let

\[ X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
we are left to solve

\[ A_2(X_1^2 - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} X_1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = -X_1^3 + \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

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So we are left with a condition about row space.
results

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Seemed easier to come up with compatible sets rather than incompatible sets. For instance: for

\[ X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_2, X_3 \}, \{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_2, X_3 \}, \{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, X_2, X_3 \}

are all compatible sets.
We need to come up with $X_1$ such that the row space of

$$-X_1^3 + \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X_1 - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not contained in the row space of

$$X_1^2 - \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} X_1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
Taking

\[ X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \]
Taking

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get

\[ A_2 \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ 0 & 0 \end{pmatrix} \]
in fact, for any

\[ a \in \mathbb{C}, a \neq 0, \frac{1}{2} \]

\[ \{ \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, X_2, X_3 \} \]

is not compatible.
For any $X_1, X_2 \in M_k(\mathbb{C})$ one can find $X_3 \in M_k(\mathbb{C})$ such that:

$$\{X_1, X_2, X_3\}$$

is compatible and $Y_3 \in M_k(\mathbb{C})$ such that

$$\{X_1, X_2, Y_3\}$$

is not compatible.
Continuing Work

- instead of having $X_2 - X_3 = I$, take $X_2 - X_3$ to be an arbitrary invertible matrix

- generalize further

- use Jordan Canonical form to write $X_i$ as a matrix composed of the direct sum of an invertible matrix and a nilpotent matrix, and find suitable conditions to solve for the corresponding coefficients.
THANK YOU!