1 Notes, Day 1

On planar graphs, orientations, and contact graphs: for a planar graph where no two vertices in $\mathbb{R}^2$ have the same y-coordinate, orient any edge $uv$ so that $u \rightarrow v$ where the y-coordinate of $v$ is greater than that of $u$. In the dual, orient any edge so that they run clockwise from the edges it crosses in the original graph.

(Ex 1) A planar 2-connected graph has a drawing such that its inherited orientation has a single source/sink.

(Ex 2) The dual orientation of a graph is acyclic.

A graph’s vertices can be represented by segments in $\mathbb{R}^2$, where intersections of segments correspond to adjacency of vertices. It is NP-complete to decide whether a partial representation of segments corresponding to a subgraph of $V(G)$ can be completed.

(Ex 3) A planar bipartite graph is a contact graph of vertical and horizontal segments.

(Ex 4) Find some non-trivial condition on the partial exponentiation to make the above problem P-time solvable.

2 Notes, Day 2 (Robert Samal)

A tournament is a directed graph which without orientation is equivalent to $K_n$.

(Ex 1) Does there exist a tournament such that for every $v_1, v_2 \in V(G)$, there exists $u$ such that $u$ beats $v_1, v_2$?

Yes: consider $\Gamma(\mathbb{Z}_7, \{1, 2, 4\})$, where $\Gamma(S, D)$ is the Cayley graph $V(G) = S$, and $u - v \in D \implies u$ beats $v$ in $G$).

Theorem 1. There exists a tournament such that for every $\{v_1, \ldots, v_k\} \in V(G)$, there exists $u$ such that $u$ beats $\{v_1, \ldots, v_k\}$.

Proof. Choose a random graph on $n$ vertices for sufficiently large $n$. The number of $k$-subsets is $\binom{n}{k}$, while the non-existence of a vertex $u$ beating every element in that $k$-subset is $(1 - 1/2^k)^{n-k}$. It follows that the random graph $G_{n, 1/2}$ has probability at most $\binom{n}{k}(1-1/2^k)^{n-k} \sim n^k (1-\varepsilon)^n \xrightarrow{n \to \infty} 0$.

Explicit construction, consider $p = 4k + 3$ prime, $\Gamma(\mathbb{Z}_p, \{i^2 : i \in \mathbb{Z}_p \setminus \{0\}\})$. \qed
Theorem 2. For all \( n \), there exists a tournament on \( n \) vertices and \( \geq n!/2^{n-1} \) directed Hamiltonian paths.

Proof. Linearity of expectation: \( \mathbb{E}[\text{Hamiltonian paths in } K_n] \) is the number of ordered sequences of \( n \) vertices multiplied by the chance they form a path in that order, with probability \( (1/2)^{n-1} \).

For every graph with \( m \) edges, there is a cut with at least \( m/2 \) edges: choose a random set \( U \subseteq V(G) \) and each edge has 1/2 probability of being in the cut (either two vertices in \( U \) or \( V(G) \setminus U \)) or each in one.

Theorem 3. Dominating sets (a set \( U \) such that \( \forall v \notin U, \exists u \in U : uv \in E(G) \)): for a probability \( 0 < p < 1 \), there exists a dominating set \( U \) of size \( \leq np + n(1-p)^{\delta+1} \); choose \( p = 1 - \sqrt[\delta+1]{1/2} \) optimal.

Proof. Make the vertices have Bernoulli distribution with \( p \) to be in \( U \); the expected size of \( U \) is \( np \). Its expected non-neighbor set is \( (1-p)^{\deg v_i+1} \leq (1-p)^{\delta+1} \); thus the expected size of the non-neighbors is \( n(1-p)^{\delta+1} \).

\[ \text{Theorem 2: } n \rightarrow \frac{n!}{2^{n-1}} \text{ Hamiltonian paths.} \]

\[ \text{Theorem 3: } \forall v \notin U, \exists u \in U : uv \in E(G), p = 1 - \sqrt[\delta+1]{1/2} \text{ optimal.} \]

3 Notes, Day 3 (Pavel Valtr)

Define a \( k \)-hole to be a convex polygon \( P \) whose vertices are the vertices of a plane drawing of a graph, such that the interior of the polygon contains no other vertices. Erdos conjectured that for every \( k \) there is a sufficiently large number of points \( m \) for which there at least one \( k \)-hole; this was shown false for \( k = 7 \). Let \( f_k(m) \) be the number of \( k \)-holes in \( P \).

Theorem 4. \( f_1(P) - f_2(P) + f_3(P) - f_4(P) + \cdots = 1. \)

For a cycle, this is equivalent to the binomial theorem. Move a point slowly and see what happens, double check when 3 points become collinear.

Theorem 5. \( f_1(P) - 2f_2(P) + 3f_3(P) - 4f_4(P) + \cdots = |P \cap \text{int} \text{conv } P| \).

Theorem 6. The number of special triangles is \( n^2 - 5n + |\text{conv } P| + 4 \); thus number of 3-holes is at least \( n^2 - 5n + 7 \).

A special triangle here is one in which we draw the triangle and a horizontal line through the point with the highest \( y \)-coordinate, and such that the two regions defined by the horizontal line and the line through the other two vertices of the triangle define a region containing no vertices. The proof involves showing that the number of special triangles is preserved under rotation; then move a point to outside of convex hull and rotate so that point appears on bottom when it is passing through lines.

4 Notes, Day 4 (Vit Jelinek)

Let \( O(x), E(x) \) be the generating function for the number of ordered partitions of an integer \( n \) into an odd number of parts and even number of parts, respectively. Then \( O(x) = \frac{x(1-x)}{1-2x} = 1 + \sum_{i=2}^{\infty} 2^{i-2} x^i \) and \( O(x) + E(x) = \frac{1}{1-2x} = 1 + \sum_{i=2}^{\infty} 2^{i-1} x^i \). We can also consider a bijection to the set of subsets of even and odd size of \([n]\).

We say a string \( \alpha > \beta \in \{0,1\}^k \) if \( \alpha \) has a higher expected chance of appearing before \( \beta \). Prove or disprove:

**Ex 1** \( \alpha > \beta \iff \mathbb{E}[T_{\alpha}] > \mathbb{E}[T_{\beta}] \) (expected waiting times).

False: say, consider 000 and 100 for the first direction, and 100 and 110 for the second direction.

**Ex 2** \( > \) is a transitive relation.

False: say, consider 001, 010, 100
5 Notes, Day 5 (Jiri Fiala)

Combinatorial embeddings and primal dual circle packing.

6 Problems

(Ex 1) For any graph $G$, evaluate $\mathcal{S} = \sum_{s: V(G) \to \{\pm 1\}} \prod_{v, v_j \in E(G)} s(v_i)s(v_j)$.

$$\sum_{s: V(G) \to \{\pm 1\}} \prod_{v, v_j \in E(G)} s(v_i)s(v_j) = \sum_{s: V(G) \to \{\pm 1\}} \prod_{v \in V(G)} s(v)^{\deg v}.$$ 

If every vertex of $G$ is even-degreed, then $\mathcal{S} = 2^{|V(G)|}$; otherwise wlog $2 \nmid \deg v_1$, so

$$\mathcal{S} = \sum_{s: v \in \{\pm 1\}} \prod_{v \in V(G), i > 1} s(v_i)^{\deg v_i} = 0.$$

(1) How large of a tournament is necessary such that every $k$-tuple is beaten?

Discussed in notes.

(2) Any tournament has an odd number of hamilton paths.

(3) Find an algorithm that outputs a cut in a given graph with at least half of the edges.

Let $U_1 \sqcup U_2 = V(G)$ be a random partition of $V(G)$, and now repeat the following operation: if $v \in U_i$ satisfies $\deg_{U_i} v > \frac{1}{2} \deg v$ for $i = 1, 2$, then move $v$ into the other set $U_{3-i}$. This algorithm terminates because $|\delta(U_1)|$ is a strictly increasing invariant bounded by $|E(G)|$, and at termination we have that

$$|\delta(U_1)| = |E(G)| - \frac{1}{2} \sum_{v \in V(G)} \deg v = \frac{|E(G)|}{2}.$$

(4) Let $(\Omega, 2^\Omega, P)$ be a finite probability space in which all elementary events have the same probability. Show that if $|\Omega|$ is a prime number then no two nontrivial events (distinct from $\emptyset, \Omega$) can be independent.

Any elementary event has probability $n/p$ for some integer $0 < n < p$. Independence implies there exists three events with probabilities $a/p, b/p, c/p$ satisfying $c/p = (a/p)(b/p) \implies c = \frac{ab}{p}$, contradiction since $p \nmid ab$.

(5) Determine the expected length of the initial increasing sequence of a random permutation of $[n]$. What is the limit for $n \to \infty$?

Let $L$ denote the length of the initial increasing sequence of a random permutation of $[n]$. It follows that

$$\mathbb{E}[L] = \sum_{k=1}^{n} P[L \geq k] = \frac{1}{n!} \sum_{k=1}^{n} \binom{n}{k} (n-k)! = \sum_{k=1}^{n} \frac{1}{k!}.$$
This is the partial sums of the expansion of \( e^x - 1 \) at \( x = 1 \), so the limit as \( n \to \infty \) is \( e - 1 \).

(6) Determine the probability that 1 and 2 are in the same cycle of a random permutation \( \pi \) of \([n]\).

Let \( C(1) \) denote the cycle of 1 under \( \pi \). Then \( P[|C(1)| = m] = \left( \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-m+1}{n-m+2} \right) \cdot \frac{1}{n-m+1} = \frac{1}{n} \) for any \( 1 \leq m \leq n \), whence \( \mathbb{E}[2 \in C(1)] = \sum_{m=1}^{n} \frac{1}{n} \cdot \frac{m-1}{n-1} = \frac{1}{2} \).

Alternatively, bijection.

(7) We toss a fair coin \( n \times \). What is the expected number of “runs” (sets of consecutive tosses with the same result)?

Let \( X_n \) be the expected number of runs for \( n \) tosses; then we see \( X_n = X_{n-1} + \frac{1}{2} \), \( X_1 = 1 \Rightarrow X_n = \left\lceil \frac{n}{2} + 1 \right\rceil \).

(8) Show that for \( m \) large and \( n > m(\ln n + 5) \), a random mapping \( f : [n] \to [m] \) is surjective with probability at least 0.99.

By counting arguments/induction,

\[
P[f \text{ surjective}] = \frac{1}{m^n} \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} i^n \geq 1 - m \left( 1 - \frac{1}{m} \right)^n \simeq 1 - me^{-n/m} = 1 - e^{-5} > 0.99.
\]

(9) Let \( A \) be a random \( n \times n \) matrix of \( \pm 1 \) with same probability, mutually independent. Find \( \mathbb{E}[\det(A)], \mathbb{E}[\det(A)^2] \).

For the first, consider the operation which flips the sign of all elements in the first row of \( A \); this is an involution with no fixed points on \( \mathcal{M}_{n \times n}(\pm 1) \) which maps \( \det(A) \mapsto -\det(A) \). Hence \( \mathbb{E}[\det(A)] = 0 \).

For the second,

\[
\mathbb{E}[\det(A)^2] = \mathbb{E} \left[ \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)^2 \right]
= \mathbb{E} \left( \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}^2 \right) + 2 \cdot 0 = \binom{n}{2}.
\]

(10) The \( n \)-coupon collector problem.

The expected number of coupons to collect is \( n \sum_{i=1}^{n} \frac{1}{i} \); one gains the \( n \)th new coupon with probability \( \frac{n-m+1}{n} \), and use a geometric distribution for each new coupon.

(11) For \( d \) sufficiently large, there is a set \( S \) in \( d \)-dimensional Euclidean space with more than \( 2d - 1 \) points s.t. all angles determined by triples of points of \( S \) are smaller than \( \pi/2 \).
Here is an argument that should give a much stronger bound: consider the $2^d$ vertices of a $d$-dimensional hypercube; any triple of vertices defines angles which are at most 90°. Given three vertices, wlog one at the origin, then the dot product of the other two vertices must be 0 to form a right angle at the origin. Equivalent, for each coordinate, we either assign 1 to the coordinate of the second vertex, the third vertex, or neither; hence there are $3^d$ such possibilities. There are $2^d$ vertices and hence $6^d / (2^d 3^d) \approx (3/4)^d$ probability of picking a triplet with a right angle. So the chance that choose $2^d - 1$ of those vertices with no right angle is $(1 - (3/4)^d)^{2^d - 1} \geq 1 - (2d - 1)(3/4)^d \to 1$ by Bernoulli’s inequality etc.

(12) Any 4-uniform hypergraph with 14 edges is 2-colorable.

(13) Let $H = (V, E)$ be a $r$-uniform hypergraph. Show that there exists an $r$-coloring of $V$ such that at least $(r!/r^r)|E|$ edges have all $r$ colors.

(14) For a permutation $\pi : [n] \to [n]$ let $L(\pi)$ be the longest increasing subsequence of $\pi$.

(15) Any 4-uniform hypergraph with 14 edges is 2-colorable.

(17) Let $G_{n,p}$ be a random graph on $n$ vertices with edge probability $p$. Show that if $p \in (0, 1)$ is independent of $n$ then $G_{n,p}$ is connected almost surely. What is the smallest function $p(n)$ for which you can still show that $G_{n,p(n)}$ is connected almost surely?

Define $U_k = \{ v_1, v_2, \ldots, v_k \}$, so $U_n = V(G)$. Let $\delta(U)$ be the cut of $U$ (e.g., number of edges between $U$ and $V(G) \setminus U$). Then, $G_{n,p}$ is connected if and only if $|\delta(U_i)| > 0$ for $i = 1, 2, \ldots, n - 1$. The probability that $P[|\delta(U_i)| = 0] = (1 - p)^{(n-i)}<(1 - p)^{n-1}$. Thus,

$$P[G_{n,p} \text{ connected}] \geq (1 - P[|\delta(U_i)| = 0])^{n-1} \geq 1 - (n - 1)(1 - p)^{n-1}$$

by Bernoulli’s inequality. For any $\varepsilon > 0$ and sufficiently large $n$ we have $(1 - p)^{n-1} < \frac{\varepsilon}{n-1}$, so $G_{n,p}$ is “usually” (almost surely?) connected. This argument still works as long as $p(n) > 1 - \left( \frac{\varepsilon}{n-1} \right)^{1/(n-1)}$. 